THE DERIVE - NEWSLETTER #100

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THE BULLETIN OF THE

DERIVE

USER GROUP

+ CAS-TI

Contents:

December 2015

DUG Member Rolf Metzger sent a very nice sequence of equivalence operations which results in a – late – wish for all of you. Many thanks Rolf.

$$y = \frac{\log_e\left(\frac{x}{m} - sa\right)}{r^2}$$

$$yr^2 = \log_e\left(\frac{x}{m} - sa\right)$$

$$e^{yr^2} = \frac{x}{m} - sa$$

$$me^{yr^2} = x - msa$$

$$me^{rry} = x - mas$$

Wolfgang Alvermann sent an article for our special issue DNL#100 which is based on pictures made in Madeira. So I had the idea to fill empty spaces in this DNL with some pictures from the flower island, Josef.

Echium candidans
Pride of Madeira
Stolz-von-Madeira

Dear DUG Members,

this is really a very special issue and a very special moment for me writing the 100^{th} letter. Since 25 years we have together experienced the development of CAS beginning with the 1^{st} versions of DERIVE (some of you even with mu-Math) up to TI-NspireCAS 4.0. We enjoyed the rise of DERIVE from a DOS program fitting on a small diskette to DERIVE 6.10 and were very sad and disappointed about its "official" end some years ago. It is not surprising for the Old-Derivians that DERIVE is still surviving in so many computer brains and so many human minds as well.

Meeting DERIVE brought a new fascination of doing and teaching mathematics in a then unknown and unimaginable way into our lives. The "next generation" made the same experience holding a TI-92 or Voyage 200 first time in their hands and learned loving this technical wonder. TI-Nspire CAS is at the moment the last station of this development and we don't know the further development in the future.

During these past 25 years we could make many friends from all over the world. Our conferences (starting in Krems 1992) gave the occasion for personal meetings. So many of us can connect DNL articles with private memories.

Let me take this DNL as an excellent example: I could meet nearly all authors at several occasions in US, UK, Germany, Austria and other places. I could not meet R. Gough and D. Halprin until now but we exchanged so many emails that we now know much from each other.

Please take the opportunity to meet again or the first time at TIME 2016 in Mexico. It will be great to say "Hello, how are you?" to you.

We can be very proud that Albert Rich and David Stoutemyer – the Fathers of DERIVE – gave the honour to contribute for this issue – not a light fare, indeed. Albert is also "Father of RUBI" and David is one of the "Fathers of the TI-92 and its successors. I am also very grateful for Peter Balyta's (Texas Instruments) kind words for our jubilee.

In this DNL you will find the second part of Rob Gough's – for him very time consuming - investigation on prime numbers followed by an article submitted by Benno Grabinger. His paper reminded me on my first years as teacher, so I could not resist to add some comments.

Wolfgang Alvermann sent a contribution for DNL#100 which was inspired by a stay on Madeira. My wife and I were there two years ago, so I felt inspired to include some flowers from there – and a DERIVE treatment of his pavements.

At last I have to thank for your friendship and loyalty over so many years. The DNL could not have existed without your great and wonderful cooperation for such a long time.

Let's start into the next twenty fives !

Best regards until #101 in 2016

Josef

Download all *DNL*-DERIVE- and TI-files from
http://www.austromath.at/dug/

The *DERIVE-NEWSLETTER* is the Bulletin of the *DERIVE* & CAS-*TI User Group*. It is published at least four times a year with a content of 40 pages minimum. The goals of the *DNL* are to enable the exchange of experiences made with *DERIVE*, *TI*-CAS and other CAS as well to create a group to discuss the possibilities of new methodical and didactical manners in teaching mathematics.

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Contributions:
Please send all contributions to the Editor. Non-English speakers are encouraged to write their contributions in English to reinforce the international touch of the *DNL*. It must be said, though, that non-English articles will be warmly welcomed nonetheless. Your contributions will be edited but not assessed. By submitting articles the author gives his consent for reprinting it in the *DNL*. The more contributions you will send, the more lively and richer in contents the *DERIVE* & CAS-*TI Newsletter* will be.

Next issue: March 2016

Preview: Contributions waiting to be published

Some simulations of Random Experiments, J. Böhm, AUT, Lorenz Kopp, GER
Wonderful World of Pedal Curves, J. Böhm, AUT
Tools for 3D-Problems, P. Lüke-Rosendahl, GER
Hill-Encryption, J. Böhm, AUT
Simulating a Graphing Calculator in *DERIVE*, J. Böhm, AUT
An Interesting Problem with a Triangle, Steiner Point, P. Lüke-Rosendahl, GER
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Cubics, Quartics – Interesting features, T. Koller & J. Böhm, AUT
Logos of Companies as an Inspiration for Math Teaching
Exciting Surfaces in the FAZ / Pierre Charland´s Graphics Gallery
BooleanPlots.mth, P. Schofield, UK
Old traditional examples for a CAS – what´s new? J. Böhm, AUT
Where oh Where is It? (GPS with CAS), C. & P. Leinbach, USA
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Tutorials for the NSpireCAS, G. Herweyers, BEL
Some Projects with Students, R. Schröder, GER
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Henon & Co; Find your very own Strange Attractor, J. Böhm, AUT
Rational Hooks, J. Lechner, AUT
Simulation of Dynamic Systems with various Tools, J. Böhm, AUT
Statistics of Shuffling Cards, Charge in a Magnetic Field, H. Ludwig, GER

and others

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A Rule-based Revolution

Organizing Mathematical Knowledge as a Rule-based Decision Tree

Albert D. Rich, Applied Logician
Hawaii Island, December 2015

I am the author of 3 implementations of the ***muLisp*** artificial intelligence development system, and coauthor of the ***muMath*** and ***Derive*** computer algebra systems. However, I am convinced my current research will be by far my most important and enduring contribution to the fields of math and computer science.

Since the beginning of recorded history, mathematicians have been amassing mathematical truths at an ever greater rate and into ever more esoteric realms. However, this vast amount of knowledge has not been systematically organized so the precise formula required to solve a particular problem can easily be found. It's like leaves scattered on a lawn, rather than organized into a heavily branched tree with each leaf attached to the appropriate limb.

It's Feasible

I am convinced much of mathematical knowledge can be organized into a rule-based decision tree that is unique and optimal. Others have proposed similar such grandiose programs, but the rule-based paradigm actually makes it possible to accomplish it. And this is not just armchair speculation.

Over my almost 40 year career implementing computer algebra systems, I am driven to the realization that rule-based systems are the optimal way to get optimal results. This is not limited to my current focus: indefinite integration. ***muMath*** and ***Derive*** are general purpose programs able to solve a broad range of problems. Throughout their development these systems gained knowledge by the addition of new rules and the generalization of existing ones.

David Stoutemyer and I began the design and implementation of ***muMath*** in the late 1970s. From the earliest versions, despite being limited to a 64 kilobyte address space, the system provided a mechanism for assigning rules to the property lists of functions and operators. When an expression was being evaluated, the property list of its top-level function or operator was searched to see if there was a rule specifically for the expression's second-level function or operator. If so, the rule was applied and the process repeated recursively until no more rules applied.

Although crude by today's standards, this two-level pattern matching mechanism was used extensively in the implementation of ***muMath***. Also its modularity and simplicity allowed users to add their own rules to the system. ***Derive*** was also heavily dependent on rules to simplify, solve and integrate expressions; but the rules were “hard wired” in programming code. Although more powerful than ***muMath***, ***Derive*** was often criticized for not allowing users to define their own rules for built-in functions and operators.

Computer hardware and software are highly ephemeral, so the mathematical knowledge in ***muMath*** and ***Derive*** will soon be lost. After development of ***Derive*** was terminated prematurely in 2005, I was determined to find a way to pass on the accumulated knowledge in such systems that would survive changing technology.

To that end, I began implementing a ***ru***le-***b***ased ***i***ntegrator, nicknamed ***Rubi***. Currently it consists of over 6200 integration rules organized into a decision tree that uniquely determines the appropriate rule to apply to a given integrand. Stored as a simple directed graph, this decision tree is what distinguishes ***Rubi*** from the program code used by its predecessors to select rules.

Indefinite integration accounts for only a tiny fraction of all mathematical knowledge. However, based on my time implementing ***muMath*** and ***Derive***, I see no reason why much of the rest of mathematics cannot be organized as a rule-based decision tree as well. Certainly much of analysis including equation solving, expression simplification, differentiation, summation, limits, etc. can be automated using this paradigm.

It's Desirable

So assuming a discipline of mathematics can be reduced to a rule-based decision tree, is it worth the considerable effort required? My experience implementing ***Rubi*** clearly indicates the answer is a resounding *yes!* The benefits include:

- While loading a rule-based system, it is easy to enclose each rule in a “wrapper” that displays the rule when it is applied and temporarily suspends evaluation so the result of the application is displayed. This is exactly what occurs in Derive 6.1, and now in Rubi, when showing the steps required to simplify an expression. Not only is this ability to show steps great for pedagogical reasons, when debugging the system it makes it relatively easy to track down errant rules.
- The ability to craft rules tailored for specific classes of problems makes possible the fine control required to produce optimal results. Whereas systems that depend on monolithic, one-size-fits-all algorithms frequently return dramatically inferior ones. For example, there are hundreds of problems in the integra-

tion test suite where the major commercial systems return incomprehensible, multi-page antiderivatives, but ***Rubi*** returns a simple, compact one involving only elementary functions.

- A rule-based system solves problems by evaluating predicate tests in a decision tree to determine the appropriate rule to apply. For a balanced tree having n rules, $\log_2 n$ test evaluations are required to select the appropriate rule. For example, since ***Rubi 4.8*** has about 6200 rules in a relatively balanced tree, it only needs to evaluate 12.6 tests on average to select an integration rule.

  To fully solve a problem several such applications may be required. For example, ***Rubi*** uses 308,301 rule applications to integrate the 55,430 problems in its test suite, for an average of 5.56 applications per problem.

  Therefore, only about 70 (12.6 x 5.56) tests are required to fully integrate a typical problem in the test suite. What's more, the tests are fast and easy to perform. For example, common in ***Rubi*** are tests to determine if an exponent is an integer, fraction or symbolic; or if the discriminant of a quadratic (b^2-4 a c) is positive, negative or of unknown sign.

  But since ***Rubi 4.8*** is slowed down by its use of pattern matching to search down a list of over 6,000 rules, it is only able to integrate expressions at roughly the same rate as ***Mathematica***'s built-in integrator. However, the forthcoming version 5 of ***Rubi*** uses the highly efficient *if-then-else* control construct instead of pattern matching to select rules. Preliminary testing indicates ***Rubi 5*** integrates expressions almost 2 orders of magnitude *faster* than ***Rubi 4.8*** or ***Mathematica***.

- Since virtually all computer algebra systems provide an *if-then-else* control construct, it will be relatively easy to port ***Rubi 5*** to a variety of such systems. Already written are programs that automatically translate *if-then-else* decision trees written in ***Mathematica*** into the syntax used by 3 other computer algebra systems.

- In a rule-based system each rule is independent in the sense that it can be added or deleted from the system without affecting the other rules. This modularity makes implementing such systems amenable to group development with individuals able to focus on their own areas of expertise.

- The decision tree of a rule-based system must provide an appropriate rule for *every* possible instance of a given form. During development, “holes” in the tree indicate exactly where new rules can and need to be discovered. Time and again during the development of ***Rubi***, filling such holes has resulted in finding hitherto unknown (at least to me) integration formulas. Thus the process of implementing a rule-based system often leads to the discovery of some exciting new mathematics.

All the integration rules, test suite problem and results demonstrating the above benefits are freely available on Rubi’s website at

http://www.apmaths.uwo.ca/~arich/

It's Real

Finally it should be noted that ***Rubi*** is an ever improving, but imperfect, model of the actual integration decision tree existing in some Platonic sense. In the same way, software simulating a hurricane is just an imperfect model of the actual hurricane blowing in the physical world. So ***Rubi*** may be just a model, but she's a darn good looking one.

Not only does an optimal integration decision tree exist, it is unique. Time and again in order to achieve optimal antiderivatives for integrands of a given form, I was forced to restructure ***Rubi***'s decision tree and/or discover new rules. Often this repetitive process severely tested my deeply held belief that an optimal set of rules for integrating expressions must exist. But eventually the process always did converge, leaving me no choice in the design of the decision tree.

Rubi provides proof-of-concept of the utility of organizing mathematical knowledge in the form of a rule-based decision tree. Hopefully it will convince others in the math and computer science communities to join the rule-based revolution, and explore whether it is applicable to their field of interest as well.

A way to make Derive and TI-Nspire identify floats

by
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Abstract

Often it is impossible to obtain an exact closed-form solution to an equation, or a definite integral, an infinite series etc., but it is possible to obtain a floating-point approximation. You do not have to give up seeking an exact value! There are several websites and free downloadable programs that, when given a floating-point constant, attempt to determine candidates for what the corresponding float-free exact constant might be. If a proposed exact constant is plausible, concise, and agrees closely with your float, then it **often is** the desired exact constant. That is for **you** to prove, disprove, or to trust or dismiss via successively higher precision comparisons. Without candidates you would not know what to prove or disprove or to compare with successively higher-precision numerical approximations.

For another project [6] I needed a *Mathematica* **function** that returned such candidate exact constants, but there is no such built-in function. Therefore I am developing my own. The purpose of this article is to explain the method so that anyone who is interested can implement one for *Derive* or TI-Nspire.

1. Introduction:

Try this: A 16-digit float approximation to

$$\sqrt{2}-\frac{2\pi}{3}$$

is 3.508608664766291. Paste this float into the input fields at the following three websites:

http://www.wolframalpha.com/

http://mrob.com/pub/ries/index.html

https://isc.carma.newcastle.edu.au/

They all recover the float-free constant! It is like unscrambling an egg. It appears to defy the second law of thermodynamics.

The results are not necessarily simplified nicely, because these programs do not use computer algebra to **find** candidates and only the Wolfram|Alpha site has computer algebra to simplify what was found. In fact, your float-free result might be implicit of the form $f[K]$ = constant, which you will then have to solve exactly for the float-free value of variable K – either manually or by computer algebra. (Moreover, without explicit indication, the last of the above three websites often omits signs, or approximates only the fractional part, or scales your input by a power of 10. It is for you to deduce that when it occurs.)

Try approximating other float-free mathematical constants taken from the answers to calculus textbooks or elsewhere. The more complicated the exact constant is, the more float digits you will need – often more than 16 digits. As an estimate, you might need several digits more than the total number of digits in the exact constant plus the number of occurrences of symbolic constants such as π, functions such as cos, and operators such as "+", including implicit multiplication, built-up fractions and raised exponents.

Moreover, often you will not get a correct result regardless of how many digits you provide because:

- The program you are using does not support a required function or you have not selected it in the setup panel,
- The program only tries expressions fitting certain **structural** patterns.
- An exact closed-form does not exist in terms of all the functions and symbolic constants ever published: Expressions are countable. Consequently an infinitesimal portion of the real numbers are expressable in closed form even using all of the functions and special constants every published. Although floats **are** countable, for significands of length 16 or more, floats are far more numerous than plausibly concise closed-form float-free constants. Therefore you should be sceptical of complicated candidates from programs that often return unrated or highly rated results for random floats.

Each of these programs can succeed where the other two fail, so do not give up if the first one does not succeed.

Also download and run the free java program from the website

http://www.xuru.org/mesearch/MESearch.asp ,

which has a particularly nice interface and gives likelihood estimates that the candidates are correct. You will also have to provide an error bound for the float, such as, for example, 1.0e-14 for the above 16-digit float example. You will also have to choose a set of basis functions, symbolic constants such as π, and a set of integers and/or rational numbers to use in constructing candidates. That website and the one for RIES nicely explain their **exhaustive search** algorithms.

2. How to generate candidates via an integer-relations algorithm:

The inspiration for my Propose [...] function was the Maple identify (...) function, whose algorithm is described in the thesis [5]. Here is the idea as realized in my *Mathematica* function: If for the input FindIntegerNullVector [$\{c_1, c_2, ...\}$] at least one of the constants $c_1, c_2, ...$ is a float, then the function returns an indication of failure or a corresponding list of integers $\{m_1, m_2, ...\}$ such that

$$m_1 c_1 + m_2 c_2 + \ ... \cong 0.$$

There are many ways to exploit this:

2.1 Generate a rational number, $\frac{m_2}{m_1} \cong F$:

For example:

In[1] := float1 = N [103/101, 16]
Out[1] = 1.019801980198020

In[2] := FindIntegerNullVector [{1, -float1}]
Out[2] = {-103, -101}

In[3] := Solve [-103*1 – 101*F, F]

Out[3] = $\left\{\left\{F \to \frac{103}{101}\right\}\right\}$

Success!

Most computer algebra systems have a more specialized function than FindIntegerNullVector [...] for approximating floats by rational numbers. However, although they are well suited to generate the best candidate with a particular maximum absolute or relative difference from a float, they are not as effective as FindIntegerNullVector for generating the best compromise between this difference and the conciseness of the candidate.

2.2 Generate a constant of the form, $\frac{m_1 c_1 + m_2 c_2}{m_3} \cong F$ with particular c_1 and c_2:

For example:

In[4] := float2 = N [(3√2 + 2π) / 6, 16]
Out[4] = 1.754304332383145

In[5] = FindIntegerNullVector [{√2, π, -float2}]
Out[5] = {3, 2, -6}

In[6] := Solve [3*√2 + 2*π – 6*F == 0, F]

Out[6] = $\left\{\left\{F \to \left[\frac{3\sqrt{2}+2\pi}{6}\right]\right\}\right\}$

Success again!

This idea can be generalized to any number of given float-free constants. This suggests using one vector containing many float-free constants. However:

- The computing time of FindIntegerNullVector increases dramatically with the number n of components in the vector argument of FindIntegerNullVector.
- To find a relation whose largest integer magnitude is about 10^k, the given float and the float arithmetic must have precision of d decimal digits somewhat greater than k n, even if many of the resulting integers are 0. Therefore including float-free constants whose integer coefficient would be 0 requires a higher-precision float than if you had used only the necessary set of float-free constants.

- Therefore it is better to better to try several input vectors whose entries rather tightly span actual examples found in textbooks and research literature – even if there is overlap in these vectors.

2.3 Generate a quadratic number :

For example:

In[7] := float3 = N [(15 + 4 √10) / 30, 16]
Out[7] = 0.9216370213557839

In[8] := FindIntegerNullVector [{1, float3, float3^2}]
Out[8] = {-13, 180, -180}

In[9] := twoCandidates = Solve [-13*1 + 180*F – 180*F^2 == 0, F]

$$\text{Out[9]} = \left\{\left\{F \to \frac{15-4\sqrt{10}}{30}\right\}, \left\{F \to \frac{15+4\sqrt{10}}{30}\right\}\right\}$$

To determine which candidate most closely matches the given float:

In[10] := N [twoCandidates , 16]
Out[10] = {{F → 0.0783629786442160}, {F → 0.9216370213557839}}

The second of two Candidates more closely matches float3.

This idea generalizes to higher-degree algebraic numbers. Beyond degree 4 you often cannot obtain an explicit result. However, it is often helpful to have a "semi-explicit" result in a form such as a *Mathematica* result Root [2 + 3 #1 + #1^6 &, 1]. The first argument of Root[..., ...] is an anonymous function, and this example means the 1st zero of the polynomial

$$2 + 3\,x + x^6.$$

(*Mathematica* has an algorithm for uniquely numbering polynomial zeros.)

Root [...] subexpressions compose, and FullSimplify [...] can simplify expressions containing them. For example,

In[11] := With [{minPoly = 4 + 7 #1 – 2 #1^3&,
FullSimplify [Root [minPoly, 1] * Root [minPoly, 2] * Root [minPoly, 3]]]
Out[11] = 2

Root [...] subexpressions can also be evaluated numerically to compare them with the input floats. For example,

In[12] := N [Root [2 + 3 #1 + #1^6 &, 1], 16]
Out[12] = 1.280125056271326

I initially disliked results containing Root [...] or the similar Maple rootOf [...] subexpressions. However, Root [...] is the best possible float-free representation when a polynomial zero is not expressible in terms of radicals. Therefore I have learned to accept Root [...]. Moreover, for degree greater than 2 they are often more concise than a result in terms of radicals.

2.4 Generate a constant of the form $\dfrac{m_1 + m_2\,\pi}{m_3 + m_4\,\pi} \cong F$

For example:

In[13] := float4 = N [(4 + 5 π) / (3 + 3 π), 16]
Out[13] = 1.725019368921002

In[14] := FindIntegerNullVector [{1, π, float4, float4*π}]
Out[14] = {-4, -5, 2, 3}

In[15] := Solve [-4*1 – 5*π + 2*F + 3π*F == 0, F]
Out[15] = $\left\{\left\{F \to \left[\dfrac{4+5\pi}{3+3\pi}\right]\right\}\right\}$

Success again!

This idea generalizes to any set of irrational float-free numerator constants and any set of such denominator constants.

2.5 Generate a constant of the form $\text{Cos}\left[\dfrac{n_2\,\pi}{n_1}\right] \cong F$:

For example:

In[16] := float5 = N [Cos [3 π / 11], 16]
Out[16] = 0.6548607339452851

Inverting the cosine then dividing by π, we compute:

In[17] := FindIntegerNullVector [{1, -ArcCos [float5] / π}]
Out[17] = {-3, -11}

In[18] := Solve [-3*1 + 11*ArcCos [F] / π == 0, F]
Out[18] = $\left\{\left\{F \to \text{Sin}\left[\dfrac{5\pi}{22}\right]\right\}\right\}$

In[19] := N [Sin [5 π / 22], 16]
Out[19] = 0.6548607339452851

This result is different from $\text{Cos}\left[\dfrac{3\pi}{11}\right]$, but equivalent!

This idea generalizes to arguments having any of the previous and following forms.

2.6 Generate a constant of the form $\text{base}_1^{\text{exponent1}} * \text{base}_2^{\text{exponent2}} \cong F$:

Log is the natural logarithm in *Mathematica*. Applying it to a product of positive float-free constants allows us to distribute it over the product. Then for arguments that are a real power of a positive base, we can further convert the exponent into a multiplier of the Log of the base. For example,

In[20] := example6 = $2^{\sqrt{2}}\, e^{\pi/6}\, \pi^{2/5}$;

In[21] := logExample6 = Log [example6]

Out[21] = $\text{Log}\left[2^{\sqrt{2}}\, e^{\frac{\pi}{6}}\, \pi^{\frac{2}{5}}\right]$

In[22] := FunctionExpand [logExample6]

Out[22] = $\frac{\pi}{3} + \sqrt{2}\ \text{Log}[2] + 2\ \text{Log}[\pi]$

This is a rational linear combination of the irrational constants $\{\pi,\ \sqrt{2}\ \text{Log}[2],\ \text{Log}[\pi]\}$. Therefore we can apply the technique of subsection 2.2 to this basis vector, and then apply the inverse of Log to the resulting float-free constant to obtain a candidate. For example:

In[23] := float6 = N [example6, 16]
Out[23] = 12.00529315765626

In[24] := FindIntegerNullVector [{π, √2 Log [2] , Log [π], -Log [float6]}]
Out[24] = {5, 15, 6, -16}

In[25] := candidate6 = Exp [Solve [5*π + 15*√2 Log[2] + 6*Log[π] – 16*F == 0, F] [[1,1,2]]]

Out[25] = $e^{\frac{5\pi+15\sqrt{2}\ \text{Log}[2]+6\text{Log}[\pi]}{16}}$

In[26] := FullSimplify [candidate6]

Out[26] = $2^{\sqrt{2}}\, e^{\frac{\pi}{6}}\, \pi^{\frac{2}{5}}$

Success again! To me, such power-product examples are the most magical.

2.7 More about FindIntegerNullVector [...]

FindIntegerNullVector [...] uses Helaman Ferguson's PSLQ integer relation algorithm, which has been named "One of the top ten algorithms of the 20th century" [2]. Descriptions of increasingly complicated but more efficient variants include [7], [3], [5], [4], then [1]. The less ambitious variants should not be difficult to implement in *Derive* or in TI-NSpire.

3. How to discriminate between likely and unlikely candidates:

3.1 Digits of agreement:

We want our float and a candidate float-free constant to be **close** to each other. Therefore we need a measure of **agreement**.

Definition: Let d be the number of digits in the significand of a float F, let C be a float-free constant, and let M = Max [$|C|$, $|F|$]. The digits of **agreement** between C and F is:

- d if M is 0 or $F - C$ is 0,
- $-\text{Log10}\left[\dfrac{|F-C|}{M}\right]$ otherwise.

Remarks:

- I use base 10 rather than base 2 logarithms because most users think in terms of decimal digits rather than bits.
- The worst case is when nonzero F == -C, making $\dfrac{|F-C|}{M}|$ == 2.0; and Log10[2.0] is approximately -0.30103. Consequently -0.30103 ≤ agreement ≤ d.

3.2 Complexity := base-10 entropy:

We want the float free candidate to be **concise**. Therefore we need a measure for the **complexity** of an expression. For consistency with information theory, my definition of Complexity is based on the entropy of an expression:

Partial definition: The **base-10 entropy** of a **multi-set of positive integers** is the sum of their base-10 logarithms.

Remarks:

- This sum is equivalent to the base-10 logarithm of the product of the integers. Therefore a mentally computable **upper bound** is the **total number of digits**. However, the total number of digits is a coarse step function. In contrast, the base-10 entropy is much smoother, providing a finer discrimination between alternative candidates.
- To include the integer 0, I define its base-10 entropy as 0.

To extend the idea of base-10 entropy to include symbolic constants such as π, operators such as "+", and functions such as Cos:

- If there were ten such symbols and they were equally likely, then it seems appropriate from an information theory standpoint to add Log10[10] = 1.0 for each occurrence of such a symbol.
- Otherwise it seems appropriate to take into account their relative likelihood of occurrence, adding a lesser amount for commonly occurring symbols such as "+" or a greater amount for rarely occurring symbols such as Bessel function J_{13}[...]. This is what I do in my Complexity[...] function, but I am still experimenting and adjusting these numbers, so I will not list specific values here.

3.3 Optimizing the agreement versus complexity tradeoff:

Notice that

- All floats are rational numbers whose denominators are a power of 2 for binary or a power of 10 for decimal implementations. Thus we can always offer as a candidate the exact rational value of the float, which has an absolute error of 0, giving the maximum possible agreement *d*. However if that is what a user wants, the built-in function for rationalizing floats usually offers it. Consequently that extreme is almost certainly not what a user seeks from my Propose [...] function.
- The candidates 0 and 1 have a minimum possible complexity of 0. A user does not need the Propose [...] function to learn that 0 is an optimal exact candidate for input 0.0 or that 1 is an optimal exact candidate for input 1.0; and a user would rightfully disdain a Propose function that always returned one or both of those candidates regardless of the input.
- Consequently what the user wants from Propose [...] is an **optimal compromise** between maximal agreement and minimal complexity.

This is an example of a **multi-objective optimization** problem:

1. According to **Occam's razor** or the **Principal of Parsimony**:
   - Given two candidates having the same agreement, we should prefer the one having lesser complexity.
   - Given two candidates having the same complexity, we should prefer the one having greater agreement.
2. **Pareto optimality** [Wikipedia] formalizes these ideas to provide a way to discard some candidates for either reason, but usually more than one candidate remains.
   - Pareto pruning could discard a candidate that is the correct limit as the precision of the numerical procedure computing the float increases toward infinity.
   - However, that is unlikely for candidates produced by FindIntegerNullVector [...], because it terminates either when the agreement is as good as is justified by the precision of the input floats or the 2-norm of the integer vector becomes larger than is justified by that precision.
   - Thus if the result has an agreement much lower than the input precision or a complexity that is almost as larger or larger than the input precision, then that result is almost certainly a bogus candidate that should be rejected.
   - Conversely, if the result has a complexity that is low compared with the agreement, then it is very promising as the limit you seek.

3. To avoid the slight risk of discarding the correct result and overcome the dilemma that Pareto pruning rarely leaves only one choice, we want an optimal way to **combine agreement and complexity** so that we can **rank** the candidates and return all of the promising candidates together with their ranking scores, discarding only those that have a negligible probability of being correct.

- We would like that ranking to have a perfect correlation of 1.0 with the likelihood that the candidate is what the float would approach with increasing precision of the numerical equation-solving, integration, summation, etc. that produced the float. In other words, we would like the combined measure to be a good **discrimination** measure.

Definition: The **agreement ratio** is the ratio of the agreement to the complexity.
Definition: The **agreement margin** is the agreement minus the base-10 entropy measure of complexity.

Remark: To be meaningful, the agreement margin requires that the agreement and the complexity have the same units. That is the reason I use a base-10 logarithm for the complexity rather than the base-2 logarithms used in information theory. In contrast, the agreement ratio does not require that the agreement and complexity have the same units.

Both of these alternatives are appealing measures that can be used to rank alternatives. I have been using the agreement margin, but more experimentation is needed.

However, neither choice is a probabilistic likelihood estimate, and neither choice will be familiar to users.[1] Consequently users will at least initially have no idea how much to trust a candidate as a function of its agreement margin. Therefore, based on my limited experience so far, my implementation also returns a **quality adjective** with each candidate, tentatively defined as follows:

Adjective	Agreement margin
Excellent	$[11, ..., \infty]$
Good	$[9, ..., 11)$
Medium	$[7, ..., 9)$
Poor	$[5, ..., 7)$
Bad	$[3, ..., 5)$
Terrible	$[-\infty, ..., 3)$

Only Bad or better candidates are retained. If there are no Bad or better candidates, then the float is returned.

For each returned candidate there is a list containing the candidate, the agreement, the base-10 entropy, the agreement margin, and the appropriate adjective.

---

[1] . MESearch can provide likelihood estimates because expressions are generated in monotonically increasing order of its complexity measure, and the number of expressions considered so far is known. In contrast, each invocation of FindIntegerNullVector [...] covers a vague large number of possible result integer vectors of varying complexity, and the predetermined structures omit many possible simpler expressions that MESearch can eventually generate.

4. Trying a sequence of models

I am naming my function Propose [...] as a warning that returned float-free constants are not guaranteed to be correct – even those with an Excellent agreement margin.

The most straightforward way to try a sequence of models is to have a function for each of the six ways to use an integer-relation function described in Section 2. Glossing over details, for each of these six functions there can be a list of lists, with each inner list being:

- a list of non-float constants for Subsections 2.1 and 2.2,
- a list of polynomial degrees such as $\{2, 3, 4\}$ for Subsection 2.3.
- two lists of non-float constants for Subsection 2.4,
- a list of function names, their real-domain ranges, their inverse function name, and an argument multiplier such as π for Subsection 2.5,
- a list of non-float constants for Subsection 2.6.

Each of the six functions in turn then loops over its top-level list, trying the model and deciding whether or not its agreement margin is worthy enough to retain the candidate.

A set of cost-effective models can be inferred from studying the exercise answers to many algebra through calculus texts – not only constant answers, but also the constants in non-constant answers. Definite integrals are a particularly good source.

One particularly effective vector for the technique of Subsection 2.2 is the logarithms of successive primes up to some modest maximum such as 7, perhaps also including 1 and/or π and/or Log[π]. Other common elements in published float-free constants include $\sqrt{2}$, $\sqrt{3}$, $\sqrt{\pi}$, and π^2, $\frac{1}{\pi}$, and $\frac{1}{\sqrt{\pi}}$.

A significant majority of published non-float constants are rational numbers, rational multiples of π, and quadratic numbers or π times quadratic numbers. Those alone would delight many users. However, it is hard to resist the temptation to include more than that.

5. The state of Propose [...]

My Propose [...] function is far from done, and I have decided that I should turn it into a website because it complements the ones listed in Section 1. If such a website is launched, I will announce it in the *Derive* Newsletter.

6. A suggestion :

Whoever is interested in implementing or benefiting from these ideas for *Derive* and/or TI CAS, I suggest that you email the editor of this newsletter so that he can connect you with each other. The more people that contribute good examples, curate them, and test prototypes, the better. (I too would appreciate unusual good published examples with citations or URLs, and my email is dstout@hawaii.edu.)

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Agapanthus Africans
African Lily
Afrikanische Liebesblume

Dear Josef,
Find below the letter from Peter, congratulating to 25 years of outstanding contributions to the development of math education. I only can second every word he says.
Merry Christmas!
Stephan

+++++++++++++++++++++++++++++++++++++++

Congratulations on the 100th issue of the Derive Newsletter. I would like to thank Dr. Josef Böhm and the entire Derive User Group (DUG) for a quarter century of active engagement and leadership around the effective and appropriate use of technology in mathematics education. Your contributions towards evolving education technology and the teaching and learning of mathematics over the years have been revolutionary. You truly have raised the bar for the advancement of mathematics education. As a result of your work, ministries and departments of education and STEM (Science, Technology, Engineering, Mathematics) leaders around the world have challenged their own thinking about how mathematics could be taught given advancements in education technology and what mathematics should be taught in order to prepare students for the advanced mathematical thinking required to solve real-world problems, become the innovators of tomorrow, and go on to make our world a better place. The DUG drove rich pedagogical discussions that triggered an avalanche of scientific research - influencing needed curriculum changes worldwide.

Texas Instruments – Education Technology values our long-standing relationship with the DUG which is based on the following shared principles:

- Ensuring a strong customer focus in all aspects of education technology development.
- Teacher professional development is the foundation for the successful adoption of education technology and the advancement of mathematics education.
- The importance of supporting teachers and students with rich lessons and activities contextualizing the effective use of technology in the teaching and learning of mathematics.

I would also like to thank the DUG for helping us define the foundational elements behind our TI-Nspire CAS technology.
We value our relationships with leaders of the DUG and with every one of the members. I especially want to thank Dr. Josef Böhm for his leadership, patience, and drive for innovation in mathematics education technology and pedagogy.
We look forward to the next quarter century of collaboration with these key mathematics education leaders.

Thank you for your contribution to the mathematics community around the world.
Best regards,
Peter

Peter Balyta, Ph.D.
President, Education Technology
Texas Instruments Incorporated

Prime Pairs & Goldbach's Conjecture (2)

Rob Gough

3.8 Creating Ω1 conditions by preventing common prime-multiple pairs

With Ω1 *E*-numbers, no prime multiple can pair with itself otherwise that prime would be present in *E*. The following analysis shows how this can be modelled.

N/p *p*-multiples in *A*	$N-\dfrac{N}{p}=N\left(\dfrac{p-1}{p}\right)$ = the number of non-*p* numbers in *A* of which $n(A)$ are prime $\dfrac{N\left(\dfrac{p-1}{p}\right)-n(A)}{N\left(\dfrac{p-1}{p}\right)}=1-\left(\dfrac{p}{p-1}\right)\hat{n}(A)$ = the probability of a non-*p* composite in the blue sector of *A*	
non-*p* numbers in this sector of *B*	N/p *p*-multiples in *B*	$N-\dfrac{N}{p}=N\left(\dfrac{p-1}{p}\right)$ = the number of non-*p* numbers in *B* of which $n(B)$ are prime $\dfrac{N\left(\dfrac{p-1}{p}\right)-n(B)}{N\left(\dfrac{p-1}{p}\right)}=1-\left(\dfrac{p}{p-1}\right)\hat{n}(B)$ = the probability of a non-*p* composite in the blue sector of *B*
$\left(1-\left(\dfrac{p}{p-1}\right)\hat{n}(B)\right)$ = the probability of a non-*p* composite in this sector of *B* $\dfrac{1}{p}\left(1-\left(\dfrac{p}{p-1}\right)\hat{n}(B)\right)$ = the fraction of *p*-multiples in *A* pairing with non-*p* multiples in *B*	$\left(1-\left(\dfrac{p}{p-1}\right)\hat{n}(A)\right)$ = the probability of a non-*p* composite in this sector of *A* $\dfrac{1}{p}\left(1-\left(\dfrac{p}{p-1}\right)\hat{n}(A)\right)$ = the fraction of *p*-multiples in *B* pairing with non-*p* multiples in *A*	$\left(1-\left(\dfrac{p}{p-1}\right)\hat{n}(A)\right)\left(1-\left(\dfrac{p}{p-1}\right)\hat{n}(B)\right)$ = the probability of composite pairs in this sector. $N-\dfrac{2N}{p}=N\left(\dfrac{p-2}{p}\right)$ = the number of non-*p* pairs in this sector. $\left(\dfrac{p-2}{p}\right)\left(1-\left(\dfrac{p}{p-1}\right)\hat{n}(A)\right)\left(1-\left(\dfrac{p}{p-1}\right)\hat{n}(B)\right)$ = the fraction of non-*p* composite pairs in this section.

The total fraction of composite pairs in this situation is:

$$\frac{1}{p}\left(1-\left(\frac{p}{p-1}\right)\hat{n}(A)\right)+\frac{1}{p}\left(1-\left(\frac{p}{p-1}\right)\hat{n}(B)\right)+\left(\frac{p-2}{p}\right)\left(1-\left(\frac{p}{p-1}\right)\hat{n}(A)\right)\left(1-\left(\frac{p}{p-1}\right)\hat{n}(B)\right)$$

and this simplifies to:

$$1-\hat{n}(A)-\hat{n}(B)+\left(\frac{p(p-2)}{(p-1)^2}\right)\hat{n}(A)\hat{n}(B)=\delta+\left(\frac{p(p-2)}{(p-1)^2}\right)\pi 0$$

So the normalized prime pair measure is:

$$\left(\frac{p(p-2)}{(p-1)^2}\right)\pi 0$$

This function reduces the prime pair measure below the $\pi 0$ value. So by preventing p-multiple pairing, we approach the prime-prime pair measure $\pi\Omega 1$. We thus have a new β^* function:

$$^{p}\beta^*_{10} = {}^{p}\left(\frac{\pi\Omega 1}{\pi 0}\right) = \frac{p(p-2)}{(p-1)^2}$$

or:

$$^{p}\pi\Omega 1 = {}^{p}\beta^*_{10} \times \pi 0 = \frac{p(p-2)}{(p-1)^2} \times \pi 0$$

where the prefix p denotes that the functions $^{p}\pi\Omega 1$ and $^{p}\beta^*_{10}$ are dependent on the choice of p. For example, with $p = 3$ and $p = 5$ then:

$$^{3}\beta^*_{10} = \frac{3}{4} \text{ and } ^{5}\beta^*_{10} = \frac{15}{16}$$

3.9 The β-function extended to real and theoretical prime measures

A direct comparison between $\pi\Omega p$ and $\pi\Omega 1$ is not possible because they are based on different E-numbers with different prime factors, but we can estimate ratios using $\pi 0$ as an intermediary as this can be based on any value of E. We define:

$$\beta_{1p} = \frac{\pi\Omega 1}{\pi\Omega p} = \left(\frac{\pi\Omega 1}{\pi 0}\right) \times \left(\frac{\pi 0}{\pi\Omega p}\right)$$

But we have

$$\beta_{10} = \left(\frac{\pi\Omega 1}{\pi 0}\right) = 0.660 \text{ and } \beta_{0p} = \left(\frac{\pi 0}{\pi\Omega p}\right)$$

hence

$$\beta_{1p} = \beta_{10}\ \beta_{0p}$$

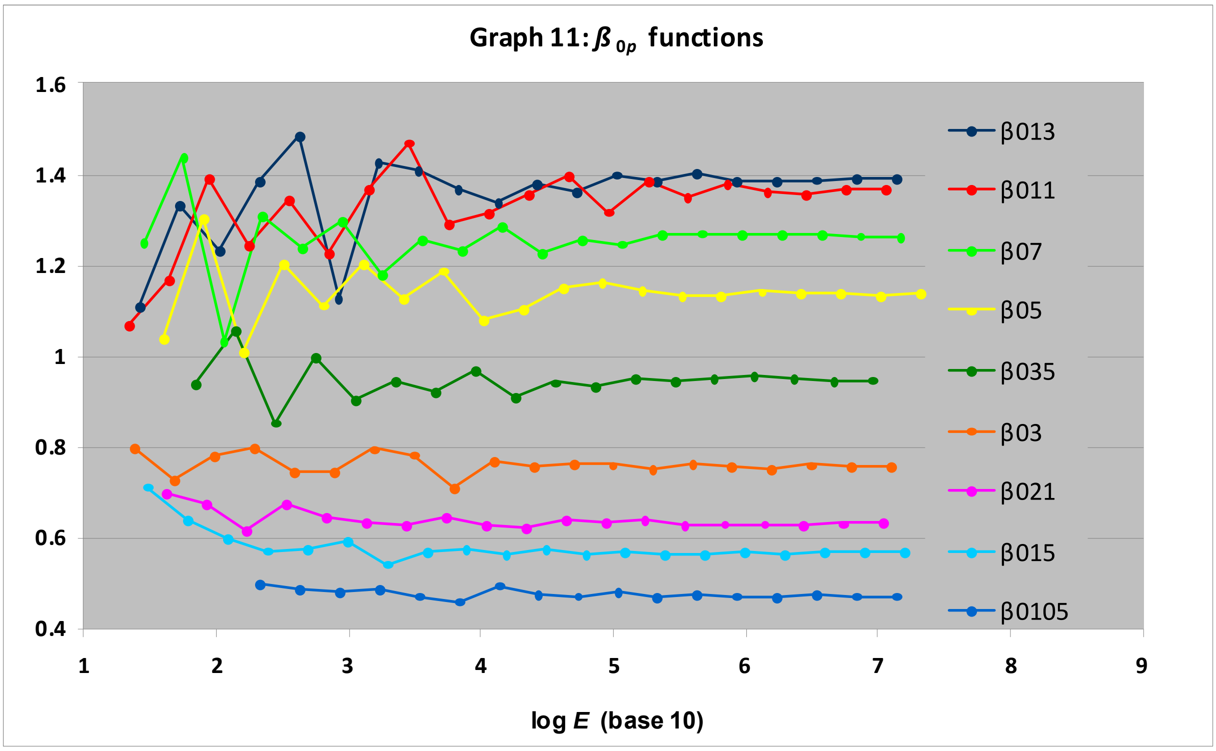
Using this nomenclature we also have the inverse function:

$$\beta_{xy} = \beta^{-1}_{yx}$$

And the ratio of a particular prime pair measure, $\pi\Omega p$ to $\pi\Omega 1$ is given by

$$\beta_{p1} = \frac{1}{\beta_{10}\ \beta_{0p}} = \frac{\beta_{p0}}{\beta_{10}}$$

Using *Derive 6* the β_{0p} functions have been calculated and the results are shown in **Graph 11**.



From this data β_{p1} has been calculated for a number of p-values and these are shown in the table below (taking $\beta_{10} = 0.66\,0$). These are based on the highest E-numbers where the β_{0p}-measures have largely settled down (see **Graph 11**). It is apparent that when p is a single prime ($\Omega = p$) β_{p1} very closely matches an integer ratio denoted β^*_{p1} where:

$$\beta^*_{p1} = \frac{p-1}{p-2}$$

and these values have been included in the table.

p	β_{0p}	$\beta_{1p} = \beta_{10} \times \beta_{0p}$	$\beta_{p1} = \beta^{-1}_{1p}$	$\beta^*_{p1} = {(p-1)}/{(p-2)}$	β^*_{p1} (dec)
3	0.758	0.500	2.000	2	2.000
5	1.138	0.751	1.331	4/3	1.333
7	1.262	0.833	1.200	6/5	1.200
11	1.366	0.902	1.109	10/9	1.111
13	1.389	0.917	1.091	12/11	1.091

This remarkable match between real data (based on prime numbers and prime-prime pairing) and simple theoretical integer ratios begs an explanation. This can be found in a modified version of the above β-formula derived from theory as:

$$\beta^*_{1p} = {}^{p}\beta^*_{10} \times \beta^*_{0p}\,.$$

From **Section 3.4** we determined the following formula for estimating the boosted prime-prime measure from a prime factor, p, in E, by identifying a common prime pairing, namely:

$$\hat{n}_{0p}(A,B) = \pi 0p = \left(\frac{p}{p-1}\right)\hat{n}_0(A,B) = \left(\frac{p}{p-1}\right)\pi 0$$

from which we identified β^*_{p0} as:

$$\beta^*_{p0} = \frac{p}{p-1}$$

and from this we get its inverse:

$$\beta^*_{0p} = \frac{p-1}{p}$$

In **Section 3.8** we found that preventing prime-pair matching, as in Ω1 numbers, leads to a scaling down from the π0 value in the direction of $\pi\Omega 1$ and:

$${}^{p}\left(\frac{\pi\Omega 1}{\pi 0}\right) = \frac{p(p-2)}{(p-1)^2}$$

and from this we have identified:

$${}^{p}\pi\Omega 1 = {}^{p}\beta^*_{10} \times \pi 0 = \frac{p(p-2)}{(p-1)^2} \times \pi 0 \quad \text{with} \quad {}^{p}\beta^*_{10} = \frac{p(p-2)}{(p-1)^2},$$

where the prefix p denotes that the function β^*_{10} is dependent on the choice of p. Combining these two results gives:

$$\beta^*_{1p} = {}^{p}\beta^*_{10} \times \beta^*_{0p} = \left(\frac{p(p-2)}{(p-1)^2}\right) \times \left(\frac{p-1}{p}\right) = \left(\frac{p-2}{p-1}\right) \text{ or by inverting } \beta^*_{p1} = \frac{\pi 1p}{\pi\Omega 1} = \left(\frac{p-1}{p-2}\right)$$

where $\pi 1p$ is the theoretical equivalent of $\pi\Omega p$ based on the real value of $\pi\Omega 1$. This hybrid measure, $\pi 1p$, is based on:

$$\pi 1p = \beta^*_{p1} \times \pi\Omega 1 = \left(\frac{p-1}{p-2}\right)\pi\Omega 1.$$

3.10 The *β*-function and multiple primes in *E*

When extended to *E*-numbers containing two prime factors (Ω*pq*) or three primes (Ω*pqr*) then the table in **Section 3.9** can be extended and the theoretical predictions suggest that the β-functions are multiplicative (see the table below).

p	β_{0p}	$\beta_{1p} = \beta_{10} \times \beta_{0p}$	$\beta_{p1} = \beta^{-1}_{1p}$	$\beta^*_{pq1} = \beta^*_{p1} \times \beta^*_{q1}$	β^*_{pq1} (dec)
3	0.758	0.500	2.000	2	2.000
5	1.138	0.751	1.331	4/3	1.333
7	1.262	0.833	1.200	6/5	1.200
11	1.366	0.903	1.109	10/9	1.111
13	1.389	0.917	1.091	12/11	1.091
15	0.5692	0.3757	2.662	$2 \times 4/3 = 8/3$	2.666
21	0.6328	0.4176	2.395	$2 \times 6/5 = 12/5$	2.4
35	0.9482	0.6258	1.598	$4/3 \times 6/5 = 8/5$	1.6
105	0.4739	0.3128	3.197	$2 \times 4/3 \times 6/5 = 16/5$	3.2

From this we get:

$$\beta^*_{pq1} = \beta^*_{p1} \times \beta^*_{q1} = \left({}^{p}\beta^*_{10} \times \beta^*_{0p} \right) \times \left({}^{q}\beta^*_{10} \times \beta^*_{0q} \right) = \left({}^{p}\beta^*_{10} \times {}^{q}\beta^*_{10} \right) \times \left(\beta^*_{0p} \times \beta^*_{0q} \right).$$

Now β^*_{pq0} has already been identified in **Section 3.5** as:

$$\beta^*_{pq0} = \beta^*_{p0} \times \beta^*_{q0}$$

Hence the inverse:

$$\beta^*_{0pq} = \beta^*_{0p} \times \beta^*_{0q}$$

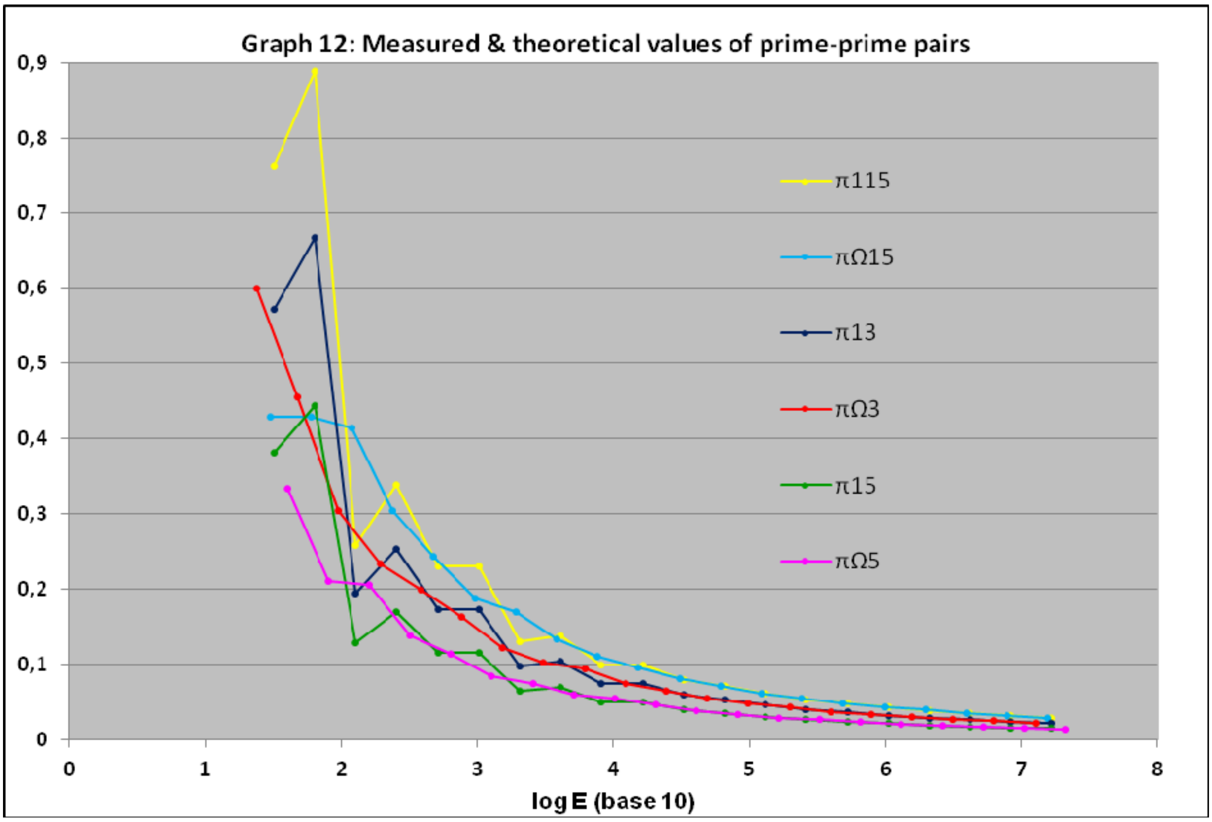
This suggests that the $\beta^*_{10}a$ functions are also multiplicative with:

$${}^{pq}\beta^*_{10} = {}^{p}\beta^*_{10} \times {}^{q}\beta^*_{10}$$

and hence:

$$\beta^*_{pq1} = {}^{pq}\beta^*_{10} \times \beta^*_{0pq}$$

Graph 12 shows some examples of real ($\pi\Omega p$) and theoretical ($\pi 1 p$) prime measures for the E-numbers containing the prime factors 3 and 5, and the composite 15. They show a strong connection between the real and theoretical measures.



3.11 The theoretical value of β_{10}

Given the multiplicative nature of ${}^{pqr}\beta^*_{10}$ as ${}^{pqr}\beta^*_{10} = {}^{p}\beta^*_{10} \times {}^{q}\beta^*_{10} \times {}^{r}\beta^*_{10}$

and that:

$${}^{3}\beta^*_{10} = \frac{3}{4} \text{ and } {}^{5}\beta^*_{10} = \frac{15}{16}$$

it would suggest that the multiplication of these numbers will converge. It therefore seems reasonable to see β_{10}^{*} as a product over all primes *p*:

$$\beta_{10}^{*} = \prod_{p=3}^{u} {}^{p}\beta_{10}^{*} = \prod_{p=3}^{u} \frac{p(p-2)}{(p-1)^2}$$

where *u* is the largest prime that needs to be considered, depending on *E*. All of the composite pairs in *E* are composed of primes in the first two thirds of *N* in the *A* group (these are the only primes that can form multiples) so *u* is the largest prime in $2N/3$. See *Appendix 2A* for further details of the calculation in *Derive*.

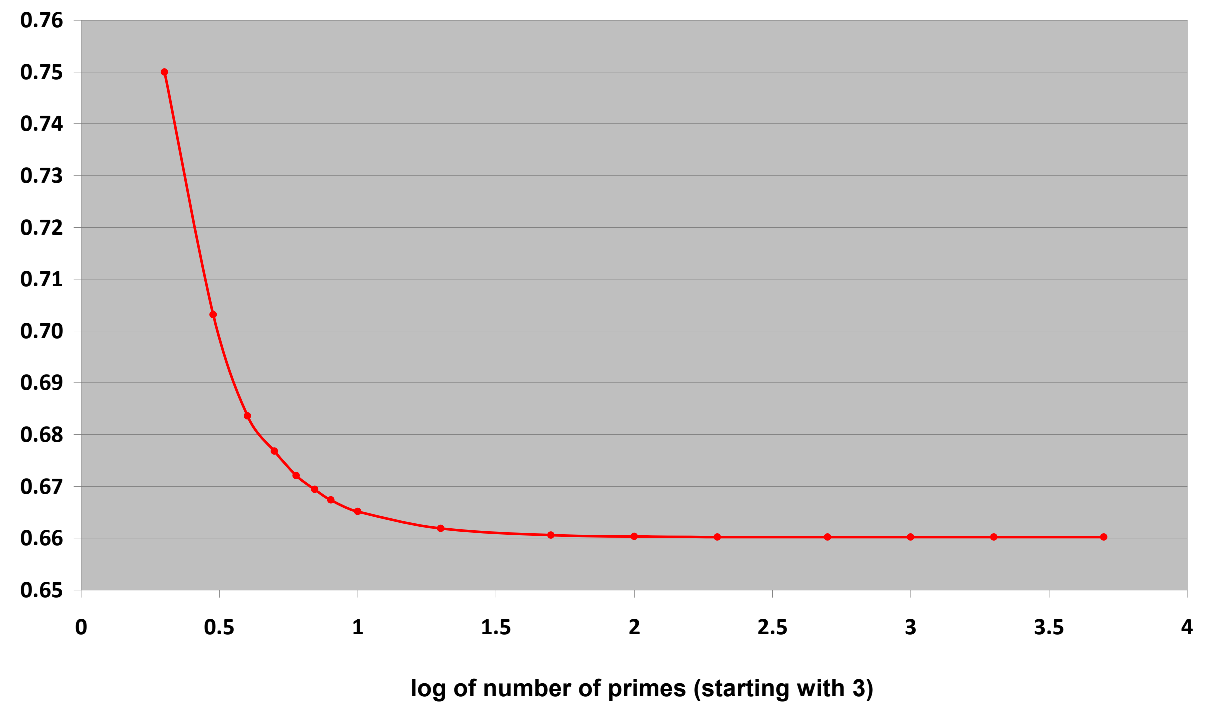
Indeed, **Graph 13** shows how β_{10}^{*} changes with the number of primes you wish to include. After about 1000 primes the value converges at 10dp to:

$$\beta_{10}^{*} = 0.6601791191$$

This theoretical value is remarkably close to the real value determined in **Section 3.7** of:

$$\beta_{10} = 0.660337319.$$

Graph 13: The theoretical measure ß*$_{10}$



3.12 The complete theoretical pair measure

Sections 3.3 to **3.6** were the first to suggest that there was a hierarchy of prime pair measures based on simple integer ratios of the prime factors of *E*. This lead to the β_{p0}^{*} -functions as a way to estimate the $\pi\Omega p$ measures based on the theoretical measure $\pi 0 p$ where:

$$\pi 0 p = \beta_{p0}^{*} \times \pi 0 = \frac{p}{p-1} \times \pi 0.$$

They were, however, totally inaccurate because they were based solely on the generalized pairing measure $\pi 0$.

A more suitable base-line measure was $\pi\Omega 1$ both because it seemed to be the lowest measure and because $\Omega 1$ numbers contain no odd prime factors which could contribute to the prime pair measure via the delta rule.

This lead to a technique developed in **Section 3.8** to remove prime multiple composites from $\pi 0$ and so reduce the prime pair measure (again via the delta rule). This, in turn, lead to the ${}^{p}\beta^{*}_{10}$-functions and:

$$^{p}\pi\Omega 1 = {}^{p}\beta^{*}_{10} \times \pi 0 = \frac{p(p-2)}{(p-1)^2} \times \pi 0$$

where the p-prefix denotes that the removal of p-multiple pairs from $\pi 0$ reduces ${}^{p}\pi\Omega 1$ in the direction of $\pi\Omega 1$ and:

$$^{p}\beta^{*}_{10} = \frac{p(p-2)}{(p-1)^2}.$$

In **Section 3.9** the relationship between the real $\pi\Omega$ measures, where:

$$\beta_{1p} = \beta_{10}\ \beta_{0p} \approx \frac{p-2}{p-1}$$

started to converge with the theoretical measures, where:

$$\beta^{*}_{1p} = {}^{p}\beta^{*}_{10} \times \beta^{*}_{0p} = \left(\frac{p(p-2)}{(p-1)^2}\right) \times \left(\frac{p-1}{p}\right) = \left(\frac{p-2}{p-1}\right).$$

And from this, a theoretical prime pair measure predictor, $\pi 1p$, was created where:

$$\pi 1p = \beta^{*}_{p1} \times \pi\Omega 1 = \left(\frac{p-1}{p-2}\right)\pi\Omega 1.$$

Still, $\pi 1p$ is a hybrid measure dependent on the calculation of $\pi\Omega 1$ with an intimate knowledge of the number and positions of all primes in A and B and how those primes pair up.

The multiplicative nature of these β-functions developed in **Sections 3.9** and **3.10** then suggested a way (**Section 3.11**) to calculate the theoretical value of β_{10} called β^{*}_{10} with the value (to 10dp) of:

$$\beta^{*}_{10} = 0.6601791191$$

This provided a strong connection between $\pi 0$ and $\pi\Omega 1$.

We are now in position to replace $\pi\Omega 1$ with a theoretical measure, called $\pi 01$, where:

$$\pi 01 = \beta^{*}_{10} \times \pi 0$$

This new measure only required knowledge of the number of primes in A and B and not on their positions or the way they paired up.

Furthermore we can now replace the old definition of the theoretical prime pair measure developed in **Sections 3.3** to **3.6** which was:

$$\pi 0p = \beta^*_{p0} \times \pi 0 = \frac{p}{p-1} \times \pi 0$$

with this more accurate version:

$$\pi 0p = \beta^*_{p1} \times \pi 01 = \beta^*_{p1} \times \beta^*_{10} \times \pi 0 = \left(\frac{p-1}{p-2}\right) \times \beta^*_{10} \times \pi 0.$$

3.13 The efficacy of the new prime pair measures

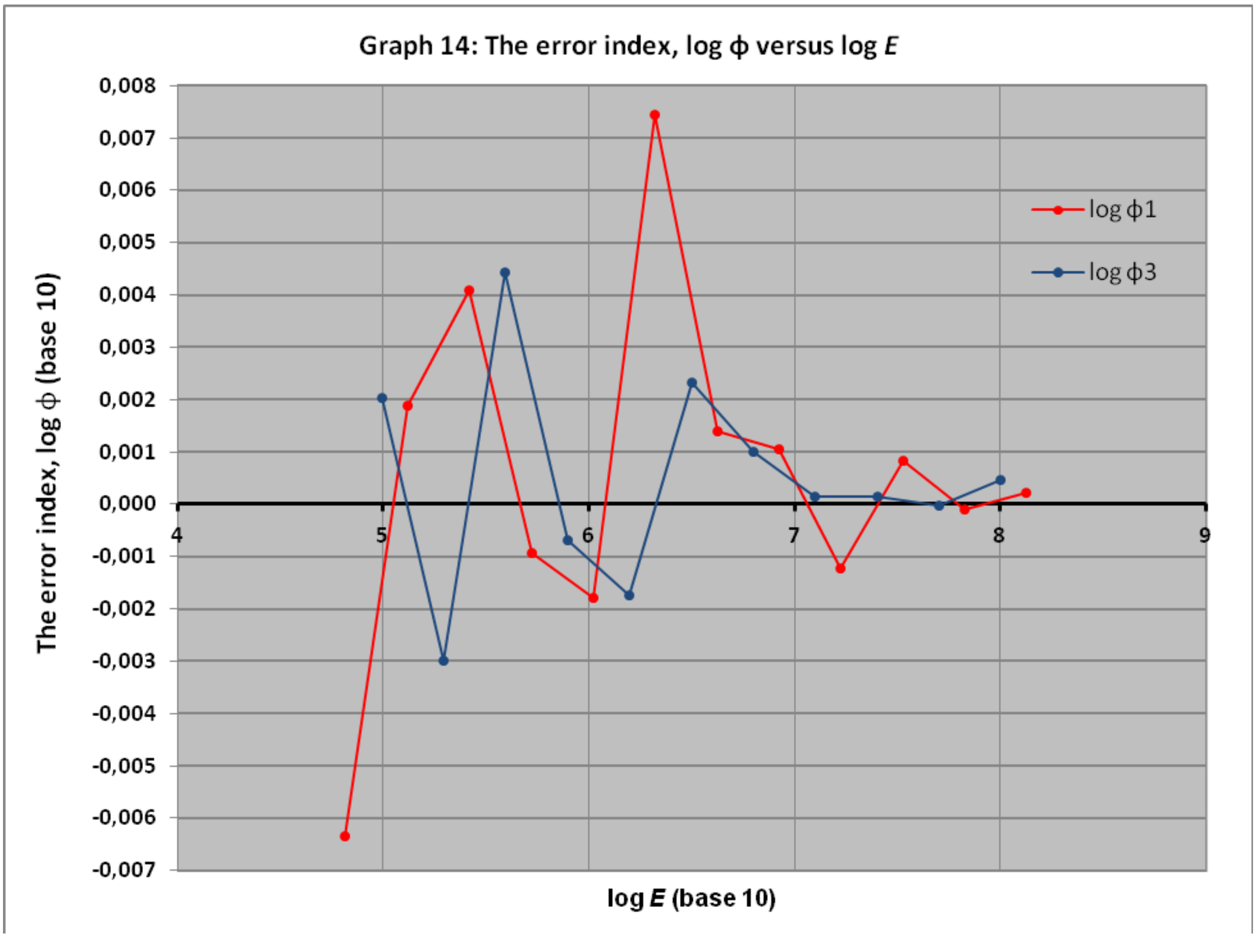
We can compare $\pi 01$ with $\pi\Omega 1$ by introducing an accuracy measure $\phi 1$ where:

$$\phi 1 = \frac{\pi 01}{\pi\Omega 1} = \frac{\beta^*_{10} \times \pi 0}{\pi\Omega 1} = \frac{0.6601791191 \times \pi 0}{\pi\Omega 1}$$

Likewise $\pi 0p$ can be compared to $\pi\Omega p$ with ϕp where:

$$\phi p = \frac{\pi 0p}{\pi\Omega p} = \frac{\beta^*_{p1} \times \beta^*_{10} \times \pi 0}{\pi\Omega p} = \frac{\left(\frac{p-1}{p-2}\right) \times 0.6601791191 \times \pi 0}{\pi\Omega p}.$$

The closer both measures are to unity the better. Taking the logarithm of this measure is akin to establishing an error index because if $\phi = 1$ then $\log \phi = 0$ and there is no error. **Graph 14** displays the error index for $\phi 1$ and $\phi 3$.



The data in **Graph 14** took about 3 days to create, but shows convincingly that the theoretical measure $\pi 01$ is a good predictor of $\pi\Omega 1$, and $\pi 03$ is a good predictor of $\pi\Omega 3$. This is especially true at higher E in the region of 100 million. The errors seem to decrease as E increases, but it would take much computing time to be sure.

An interesting question for future analysis, and requiring much greater computer time, is whether or not there exists a 'DC bias' in the error functions: their variations might decrease at higher E but tend towards a small positive or negative value. But when working at higher E values it might also be necessary to calculate β^*_{10} to more decimal places.

4. Transient effects when a new prime enters the system

Up to now we have been concerned with the long-term trends in pairing measures based on large numbers and have ignored any possible short-comings in this approach. For example, we have isolated particular prime multiples without modifying the number of remaining primes in the system – something only feasible when considering large numbers.

One common theme throughout this paper has been the sympathetic nature of the prime-prime and composite-composite measures. This sounds counter-intuitive, and it is seemingly made worse by the fact that any particular prime factor p in E is the source of p-multiples that raise the composite measure; but that prime is not then available to contribute to the prime measure that is raised as a result of the composite measure increase.

4.1 Measuring new prime multiples

As was mentioned in **Section 1.2** a new number enters the system on the left of the B-group and as E increases it moves along until it is fixed sequentially in the A-group. If that new number is a prime p then when $E = 2p$ that prime is present at the end of both the A and B groups and for that time only it pairs with itself contributing one to the total prime pair measure. That prime p has not yet formed any multiples and therefore cannot contribute to the composite pair measure increase and consequently cannot raise the prime measure via the delta rule. Indeed, when $E = 2p$ the prime measure, $\pi\Omega p$, is sometimes depressed below the $\pi\Omega 1$ level (see **Graph 2** and **4**). This is referred to as the *anomalous behaviour of new primes*.

This raises the issue of new primes entering E and how these create prime multiples as α increases in E. Consider the following table where $n_p(\overline{A},\overline{B})$ is the number of composite-composite p-pairs in E.

α	$E = p2^\alpha$	p-pairs	$n_p(\overline{A},\overline{B})$
1	2p	***A*** p ***B*** p	0
2	4p	***A*** p ***B*** $3p$	0

3	8p	***A*** *p* *3p* ***B*** *7p* *5p*	1
4	16p	***A*** *p* *3p* *5p* *7p* ***B*** *15p* *13p* *11p* *9p*	3
5	32p	***A*** *p* *3p* *5p* *7p* *9p* *11p* *13p* *15p* ***B*** *31p* *29p* *27p* *25p* *23p* *21p* *19p* *17p*	7

When $\alpha \geq 2$ the rule here is: $n_p(\overline{A}, \overline{B}) = 2^{\alpha-2} - 1$

and the normalized p-composite pair measure is:

$$\hat{n}_p(\overline{A}, \overline{B}) = \frac{2^{\alpha-2} - 1}{N}.$$

But from **Section 1.1** when $\alpha \geq 2$ we have:

$$N = \left\lceil \frac{E}{4} \right\rceil - 1 = \left\lceil \frac{p2^{\alpha}}{4} \right\rceil - 1 = p2^{\alpha-2} - 1$$

This yields:

$$\hat{n}_p(\overline{A}, \overline{B}) = \frac{2^{\alpha-2} - 1}{p2^{\alpha-2} - 1}.$$

From this we can construct a modified prime measure, $p'(\alpha)$, that takes into account the progressive development of p as α increases where:

$$p'(\alpha) = \frac{1}{\hat{n}_p(\overline{A}, \overline{B})} = \frac{p2^{\alpha-2} - 1}{2^{\alpha-2} - 1}$$

and

$$p'(\alpha) \to p \text{ as } \alpha \to \infty.$$

There would seem to be a troublesome infinity here when $\alpha = 2$, but we are not interested in p' as much as its influence on β^*_{p1}.

4.2 Developing prime-multiples & β^*_{p1}

From **Section 3.9** we have:

$$\beta^*_{p1} = \frac{p-1}{p-2}.$$

This can be modified by the results of **Section 4.1** where $\alpha \geq 2$ to give:

$$\beta^*_{p1}(\alpha) = \frac{p'-1}{p'-2} = \frac{p-1}{p-2+\dfrac{1}{2^{\alpha-2}}}.$$

The table below shows how $\beta^{*}_{p1}(\alpha)$ develops with increasing α. It also includes $\alpha = 1$ which is the same as $\alpha = 2$.

α	1	2	3	4	5
$\beta^{*}_{p1}(\alpha)$	1	1	$\frac{(p-1)}{\left(p-\frac{3}{2}\right)}$	$\frac{(p-1)}{\left(p-\frac{7}{4}\right)}$	$\frac{(p-1)}{\left(p-\frac{15}{8}\right)}$

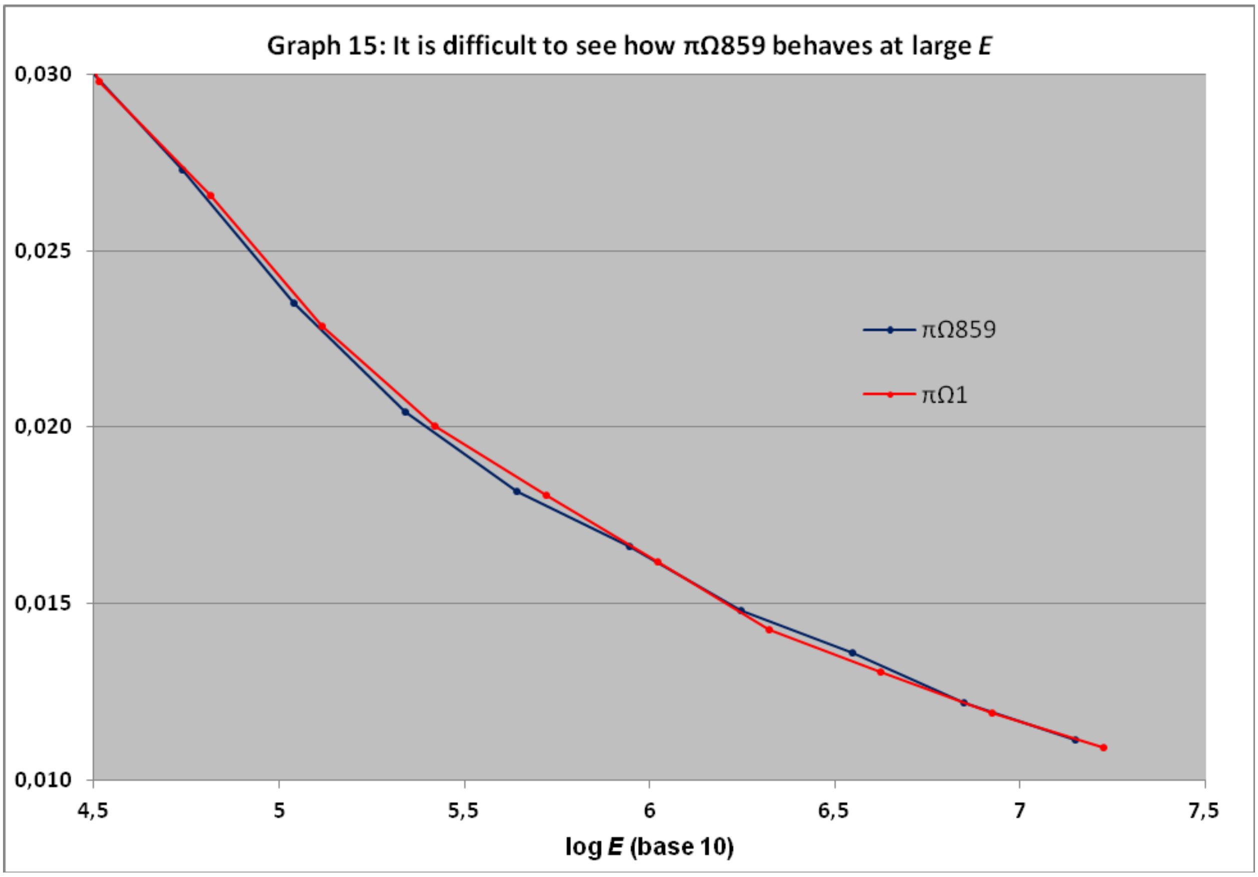
In the limit:

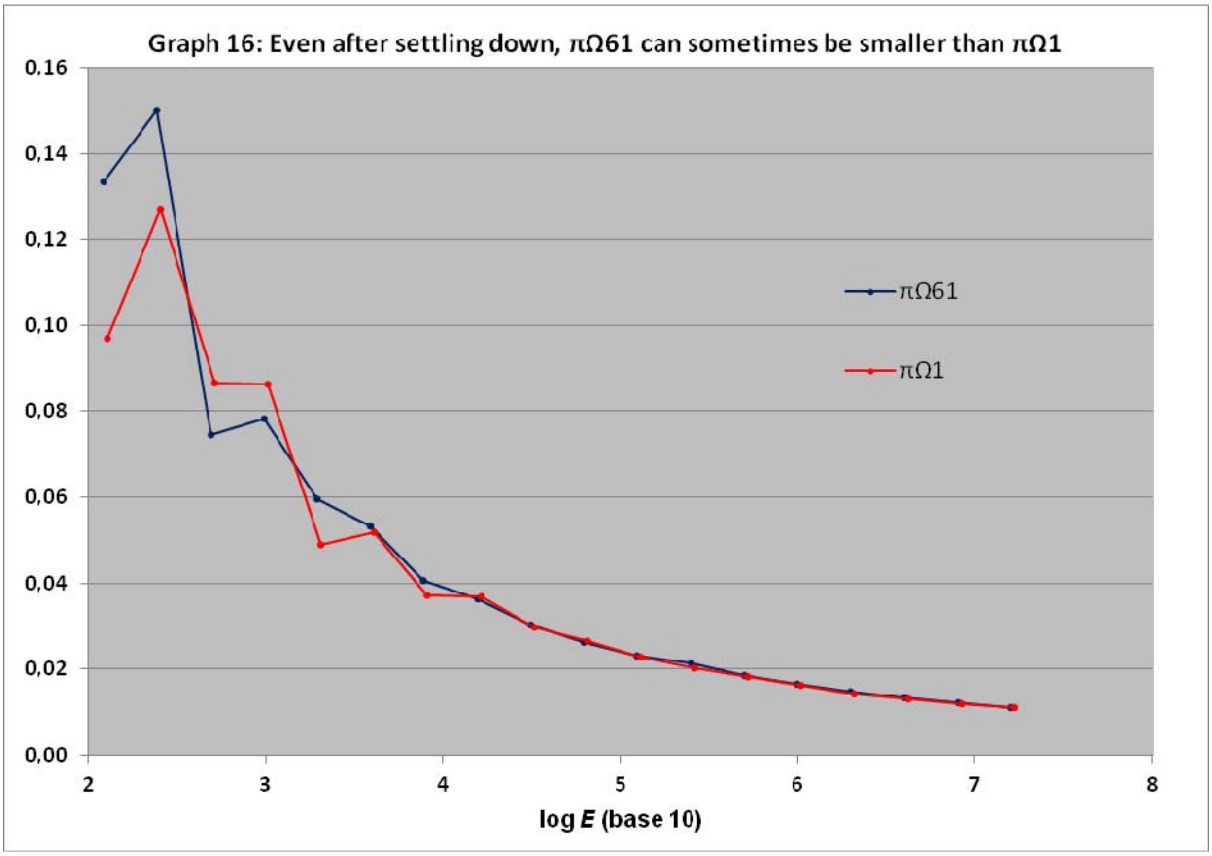
$$\beta^{*}_{p1}(\alpha) \to \left(\frac{p-1}{p-2}\right) \quad \text{as} \quad \alpha \to \infty$$

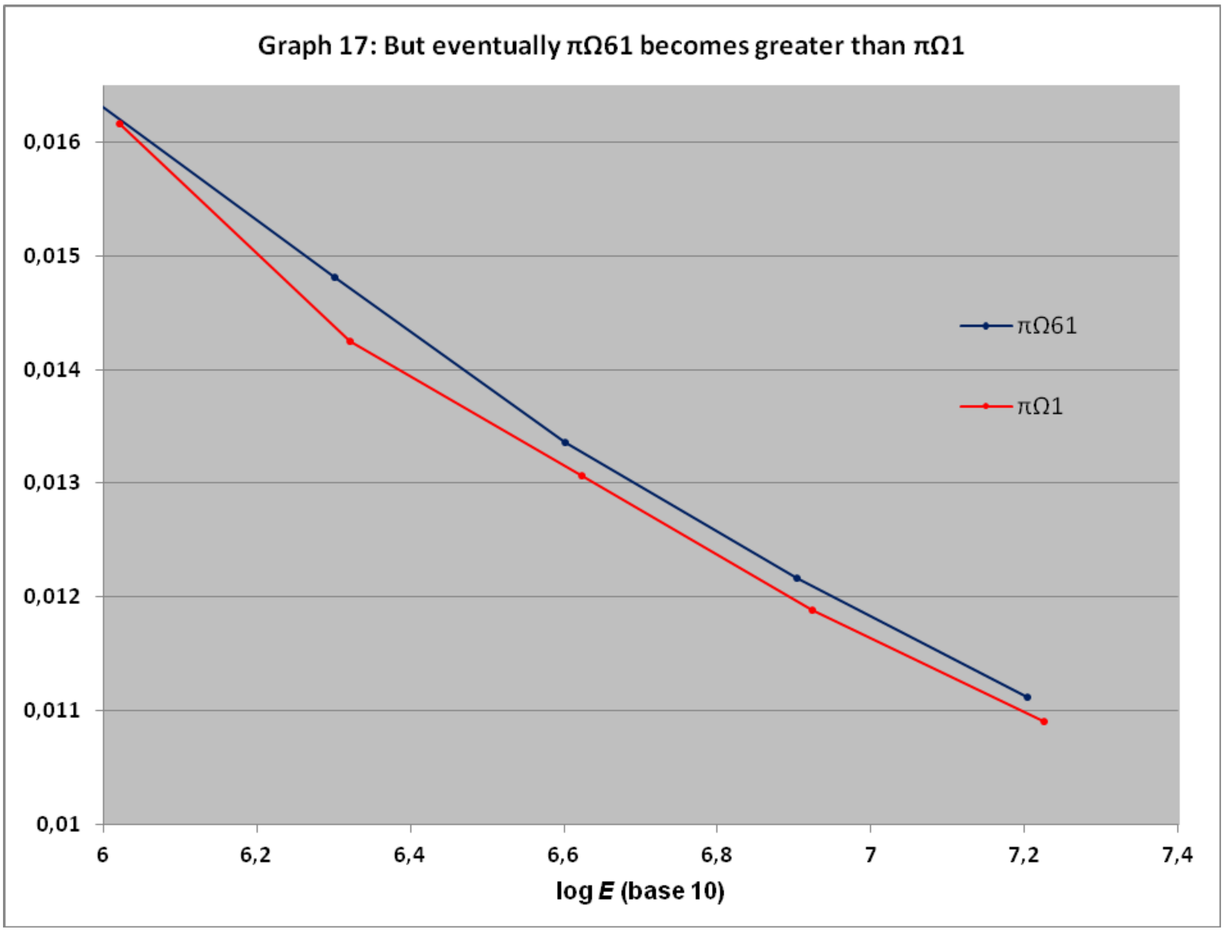
This goes some way to satisfying the observation that when a new prime p enters the system and $E = 2p$ then the $\pi\Omega p$ value is depressed, but only down to the $\pi\Omega 1$ value and not below it.

4.3 Examples of new prime entries

The advantage of the *Derive* analysis lies in the ability to plot how these primes behave at higher powers of 2. Two examples are 61 and 859 (see graphs and data). **Graph 15** compares $\pi\Omega 859$ with $\pi\Omega 1$ but it would take a lot of computing time at higher E to see if this curve eventually assumes a higher value than $\pi\Omega 1$. However, $\pi\Omega 61$ shows the trend that **Section 3.12** suggests and this is shown in **Graph 16** and **17**.




Graph 16: Even after settling down, πΩ61 can sometimes be smaller than πΩ1
0,16
0,14
0,12
0,10
0,08
0,06
0,04
0,02
0,00
2
3
4
5
6
7
8
log E (base 10)
πΩ61
πΩ1


Graph 17: But eventually πΩ61 becomes greater than πΩ1
0,016
0,015
0,014
0,013
0,012
0,011
0,01
6
6,2
6,4
6,6
6,8
7
7,2
7,4
log E (base 10)
πΩ61
πΩ1

What all these graphs show is that however low these $\pi\Omega p$ curves begin (when they enter at $E = 2p$), *they are never as low as* $\pi\Omega 1$ *will become* at higher *E*.

Meanwhile, as *E* increases the number of *p*-multiples also increases. This increases the number of composite pairs and at the same time increases the number of prime pairs by the delta rule.

Furthermore, we must take seriously the connection between the real prime pair measures via:

$$\beta_{1p} = \beta_{10}\ \beta_{0p} \approx \frac{p-2}{p-1}$$

and the theoretical prime pair measures via:

$$\beta^*_{1p} = {}^{p}\beta^*_{10} \times \beta^*_{0p} = \left(\frac{p(p-2)}{(p-1)^2}\right) \times \left(\frac{p-1}{p}\right) = \left(\frac{p-2}{p-1}\right)$$

as this connection brings to light something that would otherwise remain hidden: the prime pair measures are hierarchical and not chaotic.

5. Summary & conclusions

The results of this paper can be summarized as follows:

1. All real composite and prime measures are based on the positions of primes in the *A* and *B* groups. These measures are best considered in their normalized forms.
2. The two important measures used throughout this paper are the normalized prime-prime pairing measure $\hat{n}(A,B)$ or $\pi\Omega$ and the composite –composite pairing measure $\hat{n}(\overline{A},\overline{B})$ or $\kappa\Omega$. When looked at as *E* increases, the pattern of values seems quite chaotic, but both seem to group into two populations. The upper population is based on *E* containing the factor 3, the lower population does not contain 3. So influential is 3 on the size of prime and composite pairing measures that these two populations are quite distinct.
3. The prime-prime and composite-composite measures obey the δ-rule.
4. When $\pi\Omega$ and $\kappa\Omega$ are organised according to $E = \Omega \times 2^{\alpha}$ then order appears. For any value of Ω the values of $\pi\Omega$ eventually form a seemingly smooth curve as α increases that appears to be asymptotic towards zero.
5. The prime fractions in *A* and *B*, called $\hat{n}(A)$ and $\hat{n}(B)$, are crucial in establishing the probabilistic, generalized prime measure $\hat{n}_0(A,B)$ or $\pi 0$ and the generalized composite measure $\hat{n}_0(\overline{A},\overline{B})$ or $\kappa 0$.
6. The generalized prime and composite measures obey the δ-rule.
7. The generalized prime measure is definitely asymptotic towards zero as *E* increases because the primes thin out at higher *E*.
8. Because the generalized prime measure applies to all values of *E* it provides a link with the various real prime pair measures, which are dependent on the prime factors in *E*, through the β-functions.

9. The β-functions (real and theoretical) make a strong case for $\pi\Omega 1$ being the lowest of the prime pair measures.

10. The new theoretical prime pair measures, $\pi 01$ and $\pi 0p$, are excellent predictors of $\pi\Omega p$ without the detailed knowledge of the number and positions of all primes in A and B or how these primes pair up. As these new measures are based on $\pi 0$, which is asymptotic towards zero, then the new measures should be as well.

11. The anomalous behaviour of prime measures involving new prime entries would seem to recede at high E-values, but on this point I cannot be certain because:

    - Large computing time would be needed to observe trends
    - The theoretical analysis of **Section 4** only suggests an entry in the region of $\pi\Omega 1$ and not below it as indicated by observations.

The overall conclusion is that the above analysis strengthens the case for Goldbach's Conjecture being true (GCT) because all $\pi\Omega p$ are:

1. well behaved, especially at higher α
2. uniquely levelled by the prime factors in E, and
3. asymptotic on zero like the $\pi 0$ measure.

It also suggests that if the conjecture is false (GCF) it will fail on the $\Omega 1$ numbers because these naturally form the lowest prime-pair measures. Any new high prime entries (the seemingly anomalous primes) are only locally lower than $\pi\Omega 1$ and not lower than $\pi\Omega 1$ will become, so this should not upset the general conclusion.

In any case, when $E = 2p$, then $\pi\Omega p > 0$ because on these occasions there is a least one prime pair (itself). Therefore no new prime entry can create GCF.

Dear Josef,

I have attached the l latest version. Wasn't sure, however if you wanted the Word or pdf version so I have enclosed both.

Compared to the original version I sent you, the biggest changes are Sections 3.12 and 3.13 with the new error index graph (if you thought the other calculations took a long time, these took over 3 days!).

I have also modified Appendix 2A (see GC v9.3.pdf) to show the calculations for the error index. As you seem to like testing these out, I have also included the modified DERIVE file (Appendix 2A extended). If you know someone with a super computer, or better still a quantum computer, who could extend the computations of the error index I would be very interested to see how the errors develop.

Best wishes,

Rob

(Next page shows the second part of Rob's DERIVE file, Josef)

4. The Beta functions

4.1 A rough & inaccurate theoretical value of πΩp can be found using π0 and the beta function βsp0(Ω) (where the s stands for *) and:

#27: $$\beta sp0(\Omega) := \frac{\Omega}{\Omega - 1}$$

#28: π0p(α, Ω) := βsp0(Ω)·π0(α, Ω)

#29: Π0p(Ω, α1, α2) := VECTOR([LOG(E(α, Ω), 10), π0p(α, Ω)], α, α1, α2)

For example, with Ω3 we have:

#30: SetΠ03 := Π0p(3, 3, 22)

4.2 The real calculated Beta 10 function is defined by:

#31: $$\beta 10(\alpha, \Omega) := \frac{\pi(\alpha, \Omega)}{\pi 0(\alpha, \Omega)}$$

#32: B10(Ω, α1, α2) := VECTOR([LOG(E(α, Ω), 10), β10(α, Ω)], α, α1, α2)

SetB10 was calculated up to α26 for greater confidence in the final value but took 26 hours.

#33: SetB10 := B10(1, 5, 26)

4.3 The theoretical value for β10 is derived from pβs10(p) (where the s stands for *) and:

#34: $$p\beta s10(p) := \frac{p\cdot(p-2)}{(p-1)^2}$$

and the βs10 function is calculated using:

#35: $$\beta s10(u) := \prod_{n=2}^{u} \frac{\text{NTH\_PRIME}(n)\cdot(\text{NTH\_PRIME}(n) - 2)}{(\text{NTH\_PRIME}(n) - 1)^2}$$

4.4 The βs1p(Ω) will generate a theoretical π1p measure from πΩ1 where:

#36: $$\beta sp1(p) := \frac{p - 1}{p - 2}$$

#37: π1p(p, α, Ω) := βsp1(p)·π(α, Ω)

#38: Π1p(p, Ω, α1, α2) := VECTOR([LOG(E(α, Ω), 10), π1p(p, α, Ω)], α, α1, α2)

but Ω=1 as it is to be based based on πΩ1. So to obtain π13 and π15

#39: SetΠ13 := Π1p(3, 1, 5, 24)

#40: SetΠ15 := Π1p(5, 1, 5, 24)

5. Accuracy & the error index

5.1 The ratio of π01 to πΩ1 gives an accuracy indication, ϕ1, which if perfect is unity and:

$$\#41: \quad \phi1(\Omega,\ \alpha) := \frac{0.6601791191 \cdot \pi0(\alpha,\ \Omega)}{\pi(\alpha,\ \Omega)}$$

Taking the log of this gives the error index, Lϕ1

#42: LΦ1(Ω, α1, α2) := VECTOR([LOG(E(α, Ω), 10), LOG(ϕ1(Ω, α), 10)], α, α1, α2)

#43: SetLΦ1 := LΦ1(1, 16, 26)

5.2 The ratio of π0p to πΩp gives the accuracy indication, ϕp, which if perfect is unity:

$$\#44: \quad \phi p(p,\ \Omega,\ \alpha) := \frac{\frac{p - 1}{p - 2} \cdot 0.6601791191 \cdot \pi0(\alpha,\ \Omega)}{\pi(\alpha,\ \Omega)}$$

Taking the log of this gives the error index, Lϕp, where

#45: LΦp(p, Ω, α1, α2) := VECTOR([LOG(E(α, Ω), 10), LOG(ϕp(p, Ω, α), 10)], α, α1, α2)

#46: SetLΦ3 := LΦp(3, 3, 15, 26)

Appendix 2B: *Derive* data

log E	πΩ1	log E	πΩ3	log E	πΩ5	log E	πΩ7
1.505149978	0.285714286	1.380211241	0.6	1.602059991	0.333333333	1.447158031	0.333333333
1.806179973	0.333333333	1.681241237	0.454545455	1.903089986	0.210526316	1.748188027	0.230769231
2.107209969	0.096774194	1.982271233	0.304347826	2.204119982	0.205128205	2.049218022	0.259259259
2.408239965	0.126984127	2.283301228	0.234042553	2.505149978	0.139240506	2.350248018	0.127272727
2.70926996	0.086614173	2.584331224	0.2	2.806179973	0.113207547	2.651278014	0.117117117
3.010299956	0.08627451	2.88536122	0.162303665	3.107209969	0.084639498	2.952308009	0.089686099
3.311329952	0.048923679	3.186391215	0.122715405	3.408239965	0.075117371	3.253338005	0.080536913
3.612359947	0.051808407	3.487421211	0.102998696	3.70926996	0.059421423	3.554368001	0.061452514
3.913389943	0.037127504	3.788451207	0.094462541	4.010299956	0.055099648	3.855397996	0.052484645
4.214419939	0.036874237	4.089481202	0.073591664	4.311329952	0.045712053	4.156427992	0.042422551
4.515449934	0.029788793	4.390511198	0.064626404	4.612359947	0.03779666	4.457457988	0.038509837
4.81647993	0.026551914	4.691541194	0.054936111	4.913389943	0.032765272	4.758487983	0.032577607
5.117509926	0.022858364	4.992571189	0.048219736	5.214419939	0.029151102	5.059517979	0.028530571
5.418539921	0.020050355	5.293601185	0.042928933	5.515449934	0.026037916	5.360547974	0.024833022
5.719569917	0.018058915	5.594631181	0.037425104	5.81647993	0.023248433	5.66157797	0.02193797
6.020599913	0.016170563	5.895661176	0.033767872	6.117509926	0.020636049	5.962607966	0.019631608
6.321629908	0.014249829	6.196691172	0.030395585	6.418539921	0.018655424	6.263637961	0.017660997
6.622659904	0.013070119	6.497721167	0.02716958	6.719569917	0.016886915	6.564667957	0.015948722
6.9236899	0.011886602	6.798751163	0.024723704	7.020599913	0.015453345	6.865697953	0.014529645
7.224719895	0.010906699	7.099781159	0.022576021	7.321629909	0.014067462	7.166727948	0.013285777

log E	πΩ11	log E	πΩ13	log E	πΩ15	log E	πΩ21
1.34242268	0.6	1.414973347	0.5	1.477121254	0.428571429	1.62324929	0.4
1.643452676	0.3	1.716003343	0.25	1.77815125	0.428571429	1.924279286	0.4
1.944482672	0.19047619	2.017033339	0.2	2.079181246	0.413793103	2.225309281	0.317073171
2.245512667	0.162790698	2.318063334	0.137254902	2.380211241	0.305084746	2.526339277	0.228915663

Rob's paper contains many pages filled with Excel tables, which were used for creating the plots.

What's the Next Number?

by

Benno Grabinger, Neustadt/Weinstraße, Germany

In many intelligence tests one can find problems like this: given are the first elements of a sequence of numbers, how to continue? Let's take the numbers 1,2,4,7,11, then the test person is expected to continue with 16 because it is obvious that the differences of neighbouring numbers are 1, 2, 3, and 4. So the next number is obtained by adding 5 to 11 giving 16.

Of course, the designers of such problems expect that their own sequence rule will be recognized but in most cases they don't know that there are infinitely many possibilities to continue the given sequence. One may take 19 instead of 16 as next number – and this makes also sense. We choose (1,1), (2,2), (3,4), (4,7), (5,11) and (6,19) as nodes of an order 5 polynomial and calculate the coefficients of this polynomial. The last line of the screen shot below shows the first 10 elements of the respective sequence.

$p(x):=a\cdot x^5+b\cdot x^4+c\cdot x^3+d\cdot x^2+e\cdot x+f$ *Done*

$\text{solve}(p(1)=1 \text{ and } p(2)=2 \text{ and } p(3)=4 \text{ and } p(4)=7 \text{ and } p(5)=11 \text{ and } p(6)=19,\{a,b,c,d,e,f\})$

$$a=\frac{1}{40} \text{ and } b=\frac{-3}{8} \text{ and } c=\frac{17}{8} \text{ and } d=\frac{-41}{8} \text{ and } e=\frac{127}{20} \text{ and } f=-2$$

$p(x)|a=\frac{1}{40} \text{ and } b=\frac{-3}{8} \text{ and } c=\frac{17}{8} \text{ and } d=\frac{-41}{8} \text{ and } e=\frac{127}{20} \text{ and } f=-2$

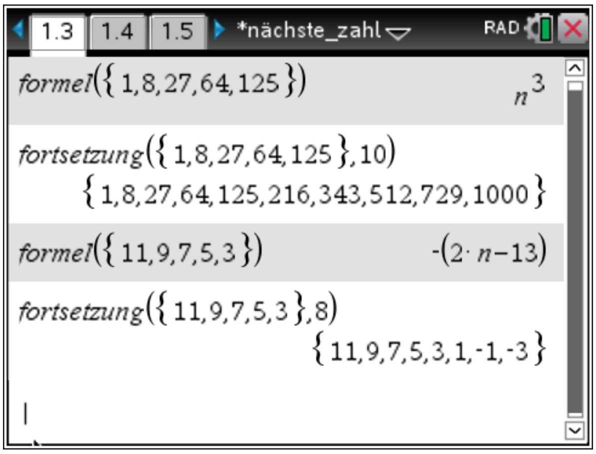
$$\frac{x^5}{40}-\frac{3\cdot x^4}{8}+\frac{17\cdot x^3}{8}-\frac{41\cdot x^2}{8}+\frac{127\cdot x}{20}-2$$

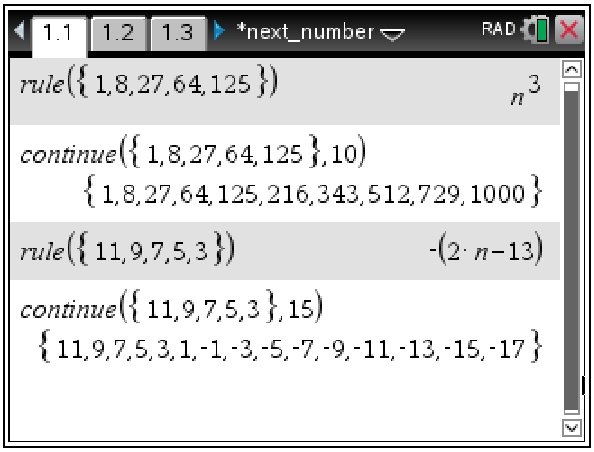
$\text{seq}\left(\frac{x^5}{40}-\frac{3\cdot x^4}{8}+\frac{17\cdot x^3}{8}-\frac{41\cdot x^2}{8}+\frac{127\cdot x}{20}-2,x,1,10\right)$

$$\{1,2,4,7,11,19,40,92,205,424\}$$

Choice of number 19 was not at all a random one but refers to Carl E. Linderholm who in his book "Mathematics Made Difficult" because of the ambiguity of the problem proposes to choose always 19 as next number completely independent of the first given numbers.

Without regarding this facts it would be great to find an algorithm which enables continuing the sequence according to the expectations of the presenter of the task. In the following we will present an algorithm which can be applied on many such number sequences appearing in intelligence tests. Besides it is a fine example for using computer algebra. The next screen shot shows how for the sequences 1,8,27,64,125 and 11,9,7,5,3 a rule can be found and then the next elements can be obtained. (German and English version are provided.)





We will avoid calculating the polynomial as done above and try another approach.

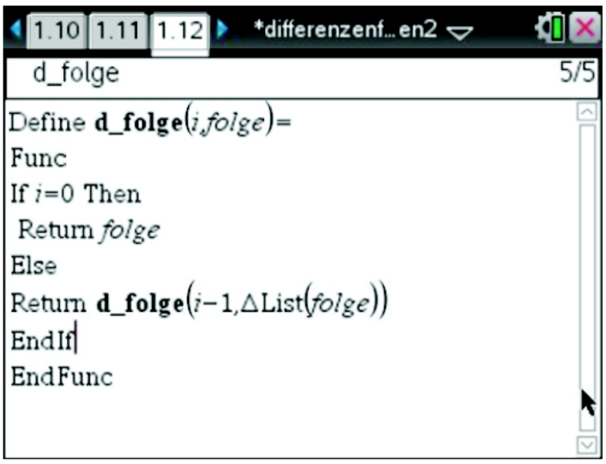
The key to this algorithm is the concept of the *sequence of differences* (SoD) – and we will focus on this concept. The 1st element of the SoD is the difference of 2nd and 1st element of the sequence, the 2nd element of the SoD is the difference of the 3rd and the 2nd element of the given sequence, etc.

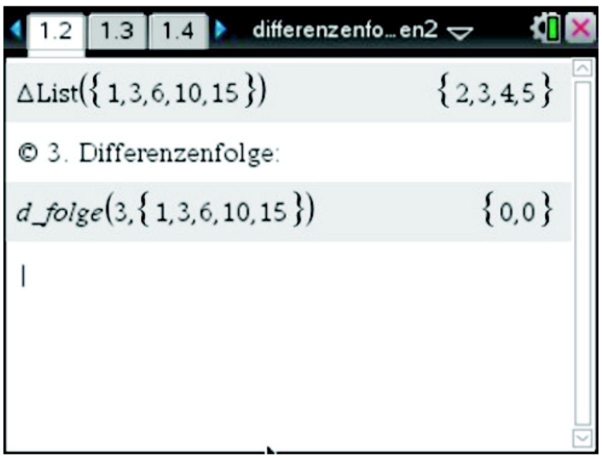
Example 1

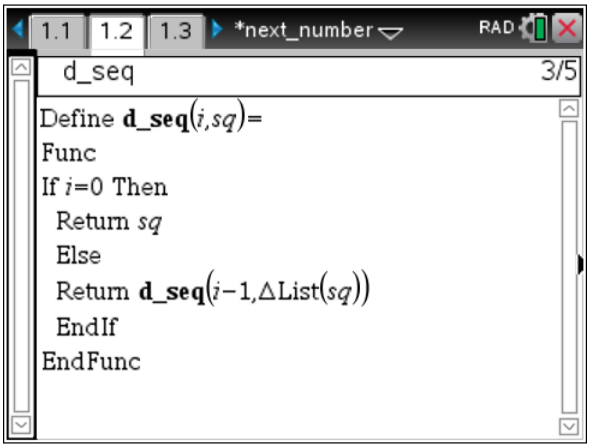
Sequence	1		3		6		10		15		...
1. SoD		2		3		4		5		...	
2. SoD			1		1		1		...		
3. SoD				0		0		...			

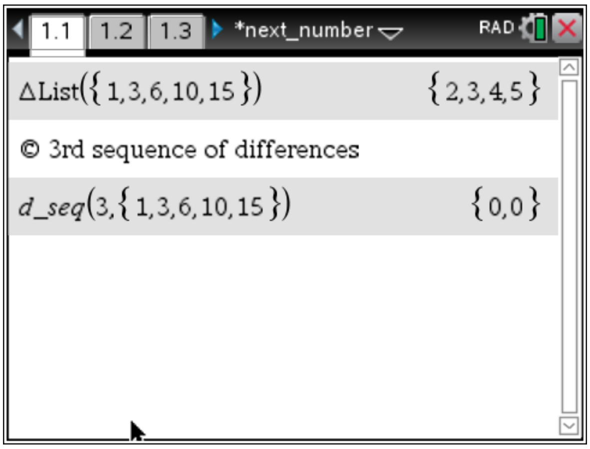
TI-Nspire gives the 1st SoD applying the function **ΔList(seq)**. SoD of higher order are created recursively. So the 3rd SoD equals the SoD of the 2nd SoD. Function **d_folge(i,folge) / d_seq(i,sq)** makes use of this principle.

Function **print** presents the rows of the differences. (See example 2.)

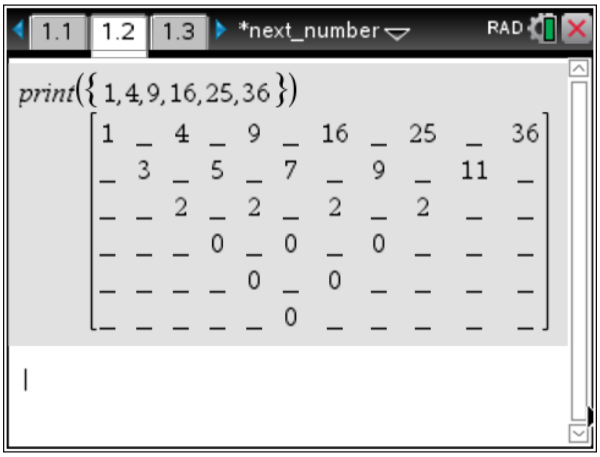


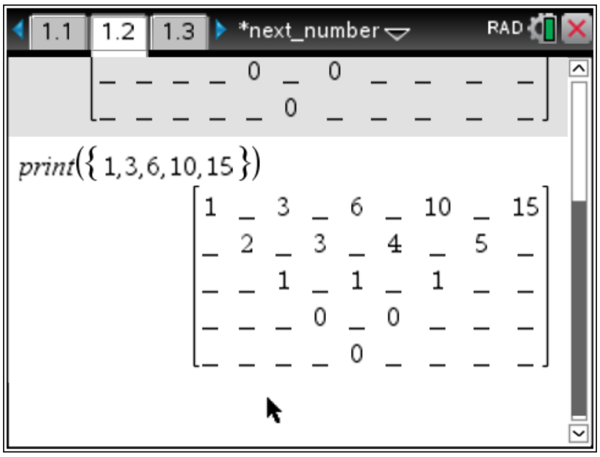






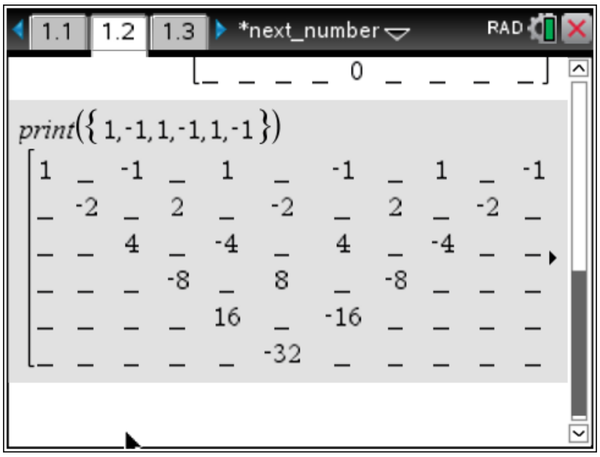
Example 2





In both examples above the SoDs develop to constant sequences. This does not occur with all sequences. Try {1,-1,1,-1,1,-1}.

But if it is the case of a constant SoD after some steps, then it is easy work to continue the given sequence. Example 3 refers to example 2.



Example 3

Sequence	1		3		6		10		15		**21**		**28**
1. SoD		2		3		4		5		**6**.		**7**	
2. SoD			1		1		1		**1**		**1**		
3. SoD				0		0		...					

Adding the next **1** in the third row (assuming that the 2^{nd} SoD remains constant) to the element 5 (left above) gives **6**. Adding 6 to 15 (again left above) gives **21**. Then start with the next **1**, leading to **7** and **28**, …

For continuing the sequence it is sufficient to know the first elements of the SoDs:

Sequence	1		
1. SoD		2	
2. SoD			1

Subsequent additions deliver the missing numbers:

Sequence	1		**3**		**6**		**10**		**15**		**21**		**28**	
1. SoD		2		**3**		**4**		**5**		**6**		**7**		
2. SoD			1		**1**		**1**		**1**		**1**			

Example 4

Let us now have a closer look to a sequence with its 3rd SoD constant. The first elements of the SoDs (SoD #0 = given sequence is included) are denoted as a, b, c, and d. Assisted by CAS the following table of the SoDs can be filled in:

	A	B	C	D	E	F	G	H	I
=									
1	a		a+b	_	a+2*b+c	_	a+3*b+3*c+d	_	a+4*b+6*c+4*d
2		b		b+c	_	b+2*c+d	_	b+3*c+3*d	_
3			c		c+d	_	c+2*d	_	c+3*d
4				d		d		d	

J	K	L	M	N	O
_	a+5*b+10*c+10*d	_	a+6*b+15*c+20*d	_	a+7*b+21*c+35*d
b+4*c+6*d	_	b+5*c+10*d	_	b+6*c+15*d	_
_	c+4*d	_	c+5*d	_	c+6*d
d		d		d	d

All is done now – more or less. One has to replace a, b, c and d by numbers for a special sequence and/or extend the table if the constant SoD appears later. I proposed to have a closer look, so let's look closer again.

Observing how often d is appearing in the cells of the table we find:

0	_	0	_	0	_	1	_	4	_	10	_	20	_	35
_	0	_	0	_	1	_	3	_	6	_	10	_	15	_
_	_	0	_	1	_	2	_	3	_	4	_	5	_	6
_	_	_	1	_	1	_	1	_	1	_	1	_	1	_

We notice that the last line consists of only ones, the next to last one is the sequence of the natural numbers, the second row is formed by the triangular numbers and the first one by the tetrahedral numbers. This scheme is well known from Pascal's Triangle:

							$\binom{n}{0}$		$\binom{n}{1}$		$\binom{n}{2}$		$\binom{n}{3}$		$\binom{n}{4}$
							1		0		0		0		0
						1		1		0		0		0	
					1		2		1		0		0		0
				1		3		3		1		0		0	
			1		4		6		4		1		0		0
		1		5		10		10		5		1		0	
	1		6		15		20		15		6		1		0
1		7		21		35		35		21		7		1	

The SoDs of $a_n = \binom{n}{3}$ give exactly the coefficients of d, while the SoDs of $a_n = \binom{n}{2}$, $a_n = \binom{n}{1}$ and $a_n = \binom{n}{0}$ deliver the coefficients of c, b, and a, respectively.

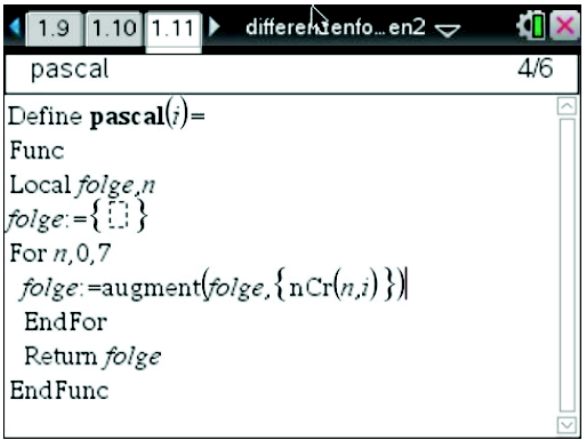
0	_	0	_	0	_	1	_	4	_	10	_	20	_	35
_	0	_	0	_	1	_	3	_	6	_	10	_	15	_
_	_	0	_	1	_	2	_	3	_	4	_	5	_	6
_	_	_	1	_	1	_	1	_	1	_	1	_	1	_

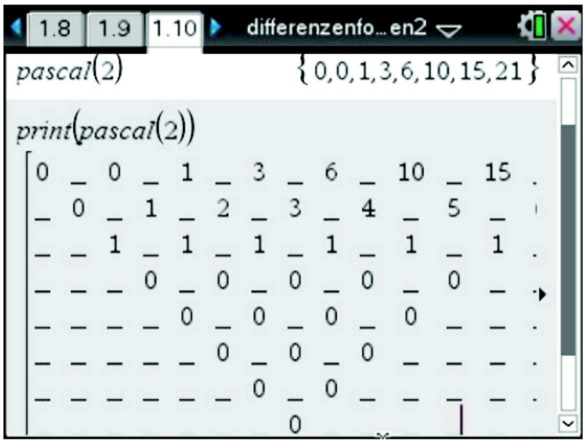
0	_	0	_	1	_	3	_	6	_	10	_	15	_	21
_	0	_	1	_	2	_	3	_	4	_	5	_	6	_
_	_	0	_	1	_	1	_	1	_	1	_	1	_	1
_	_	_	0	_	0	_	0	_	0	_	0	_	0	_

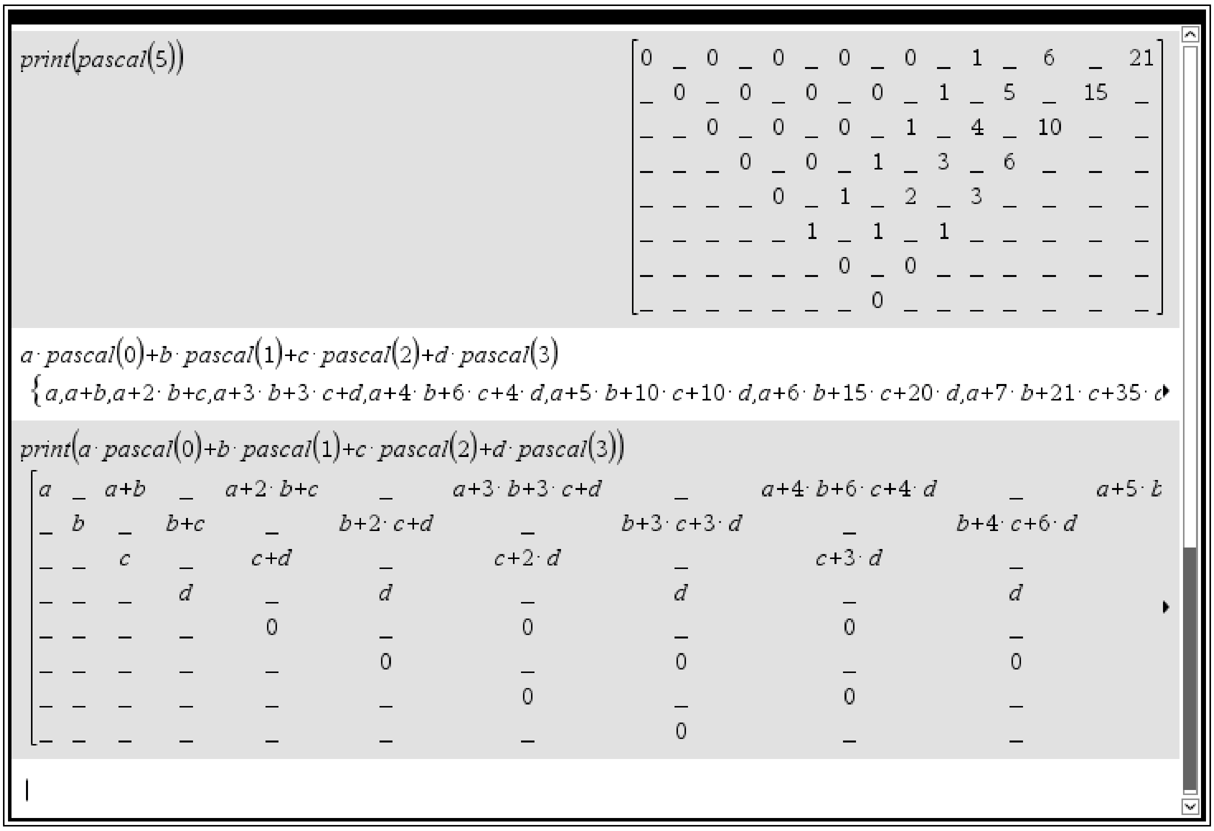
0	_	1	_	2	_	3	_	4	_	5	_	6	_	7
_	0	_	1	_	1	_	1	_	1	_	1	_	1	_
_	_	0	_	0	_	0	_	0	_	0	_	0	_	_
_	_	_	0	_	0	_	0	_	0	_	0	_	_	_

1	_	1	_	1	_	1	_	1	_	1	_	1	_	1
_	0	_	0	_	0	_	0	_	0	_	0	_	0	_
_	_	0	_	0	_	0	_	0	_	0	_	0	_	_
_	_	_	0	_	0	_	0	_	0	_	0	_	_	_

For confirming these considerations one can apply the print(sequence) function on the sequences $\binom{n}{i}$. These sequences are created using the Nspire-function pascal(i) which generates the first 8 elements.





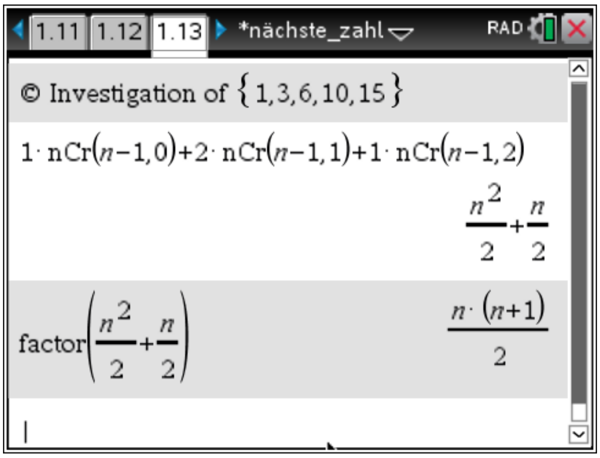


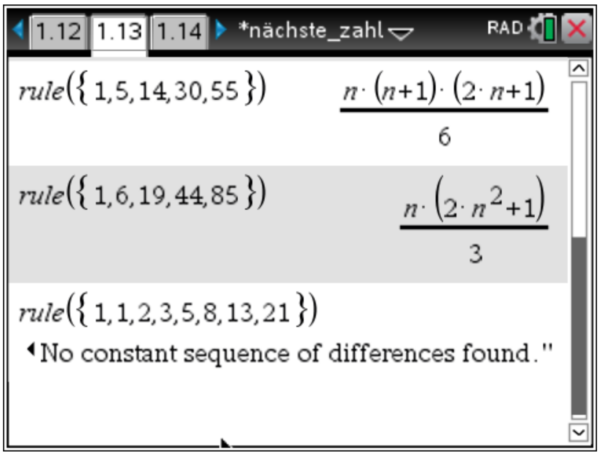
In example 1 we investigated the sequence 1,3,6,10,15,... (Triangular numbers). Its second SoD is constant. The first elements of the SoDs are 1, 2, 1.

So we have for the sequence the following expression (formula, rule):

$$1\cdot\binom{n}{0}+2\cdot\binom{n}{1}+1\cdot\binom{n}{2}.$$

We have to take into account that numbering of the elements – like in Pascal's Triangle – starts with zero. As we are used to start numbering with one we have to replace n by n-1, i.e. to take $1\cdot\binom{n-1}{0}+2\cdot\binom{n-1}{1}+1\cdot\binom{n-1}{2}$ (left screen below).





All the singular steps of calculation as forming the sequences of differences, searching for a constant SoD and finally generating the expression for the sequence formula (rule) can be collected in one function: **formel** (German version) or **rule** (English version). This function returns an expression for the elements of the sequence provided that a constant SoD can be found. This is only then the case if the sequence is described by a polynomial function. The right screen above shows the result of this function applied on the sequence of the sum of square numbers, on the sequence of the octahedral numbers and on the Fibonacci sequence.

Josef Böhm: Remarks on Benno's contribution

When I read Benno's contribution I remembered my first years as teacher. It was in the late 60ies when I – according to the then curriculum – had to teach "arithmetic sequences of higher order". And I found a chapter on this issue in a 3-volumes book which had purchased from a "bouquinist" in Amsterdam some years earlier (see page 45). This was exactly the stuff presented by Benno above (without treating the connection to the binomial coefficients). But we also considered the case when there was no constant SoD found and only one number remained in the last row. I will take the Fibonacci Sequence as an example {1,1,2,3,5,8} and I assume that 19 is the next number? Can I be right? Let's find the table of the SoDs (black numbers below). We could apply **print**.

Sequence	1	1	2	3	5	8	19	**71**	**253**
1. SoD	0	1	1	2	3	11	**52**	**182**	
2. SoD	1	0	1	1	8	**41**	**130**		
3. SoD	-1	1	0	7	**33**	**89**			
4. SoD	2	-1	7	**26**	**56**				
5. SoD	-3	8	**19**	**30**					
6. SoD	11	**11**	**11**						

Now we will assume that the last row is the constant SoD with the difference 11. So we can create the next elements of the given sequence from backwards. First we will obtain the red numbers giving 71 (the red numbers) and then 253 (the blue numbers).
I adapt Benno's function – it is really a very small change! – and it works.

And there was another thing I had in mind. Continuing the table of SoDs in the other direction (upwards) we will receive the sequence of partial sums of the respective series. The given sequence is the first SoD of the partial sums. Having obtained this sequence it is no problem to find the formula for the respective series, too.

Series		**1**	**4**	**10**	**20**	**35**	**46**
Sequence	1	3	6	10	15	21	...

Benno agreed to change his function in order to make it more general and to add the "series part", too.

See some examples (using the English function names):

© Sequence with a constant SoD after some steps:

$sq1:=\{3,5,9,15,23,33\}$ — $\{3,5,9,15,23,33\}$

$print(sq1)$

$$\begin{bmatrix} 3 & \_ & 5 & \_ & 9 & \_ & 15 & \_ & 23 & \_ & 33 \\ \_ & 2 & \_ & 4 & \_ & 6 & \_ & 8 & \_ & 10 & \_ \\ \_ & \_ & 2 & \_ & 2 & \_ & 2 & \_ & 2 & \_ & \_ \\ \_ & \_ & \_ & 0 & \_ & 0 & \_ & 0 & \_ & \_ & \_ \\ \_ & \_ & \_ & \_ & 0 & \_ & 0 & \_ & \_ & \_ & \_ \\ \_ & \_ & \_ & \_ & \_ & 0 & \_ & \_ & \_ & \_ & \_ \end{bmatrix}$$

$continue(sq1,10)$ — $\{3,5,9,15,23,33,45,59,75,93\}$

$rule(sq1)$ — n^2-n+3

$\mathrm{seq}(n^2-n+3,n,1,10)$ — $\{3,5,9,15,23,33,45,59,75,93\}$

$serie(sq1,12)$ — $\{3,8,17,32,55,88,133,192,267,360,473,608\}$

$rule\_series(sq1)$ — $\dfrac{n\cdot(n^2+8)}{3}$

$\mathrm{seq}\left(\dfrac{n\cdot(n^2+8)}{3},n,1,14\right)$ — $\{3,8,17,32,55,88,133,192,267,360,473,608,767,952\}$

English function names	**Deutsche Funktionsbezeichnungen**
rule(sequence)	formel(folge)
continue(sequence, n)	fortsetzung(folge,n)
poss(sequence)	möglich(folge)
d_seq(sequence)	d_folge(folge)
print(sequence)	print(folge)
serie(sequence,n)	reihe(folge,n)
rule_series(sequence)	formel_reihe(folge)
pascal(i)	pascal(i)

© Now let's take the Fibonacci Sequence

$sq2:=\{1,1,2,3,5,8,13\}$ $\quad\{1,1,2,3,5,8,13\}$

$print(sq2)$

$$\begin{bmatrix} 1 & \_ & 1 & \_ & 2 & \_ & 3 & \_ & 5 & \_ & 8 & \_ & 13 \\ \_ & 0 & \_ & 1 & \_ & 1 & \_ & 2 & \_ & 3 & \_ & 5 & \_ \\ \_ & \_ & 1 & \_ & 0 & \_ & 1 & \_ & 1 & \_ & 2 & \_ & \_ \\ \_ & \_ & \_ & -1 & \_ & 1 & \_ & 0 & \_ & 1 & \_ & \_ & \_ \\ \_ & \_ & \_ & \_ & 2 & \_ & -1 & \_ & 1 & \_ & \_ & \_ & \_ \\ \_ & \_ & \_ & \_ & \_ & -3 & \_ & 2 & \_ & \_ & \_ & \_ & \_ \\ \_ & \_ & \_ & \_ & \_ & \_ & 5 & \_ & \_ & \_ & \_ & \_ & \_ \end{bmatrix}$$

$continue(sq2,10)$ $\quad\{1,1,2,3,5,8,13,29,85,247\}$

$rule(sq2)$ $$\frac{5\cdot n^6-123\cdot n^5+1205\cdot n^4-5925\cdot n^3+15350\cdot n^2-19152\cdot n+9360}{720}$$

$$\mathrm{seq}\left(\frac{5\cdot n^6-123\cdot n^5+1205\cdot n^4-5925\cdot n^3+15350\cdot n^2-19152\cdot n+9360}{720},n,1,12\right)$$

$\{1,1,2,3,5,8,13,29,85,247,640,1475\}$

$serie(sq2,12)$ $\quad\{1,2,4,7,12,20,33,62,147,394,1034,2509\}$

$rule\_series(sq2)$ $\quad n\cdot(5\cdot n^6-126\cdot n^5+1274\cdot n^4-6510\cdot n^3+17885\cdot n^2-23604\cdot n+16116)$

The two screens on this page demonstrate the use of the adapted (= generalized) functions. So, for example I can insist that 19 will follow 1, -2, 3, -4 and 5 in a row and then prove my assertion.

$$\mathrm{seq}\left(\frac{n\cdot(5\cdot n^6-126\cdot n^5+1274\cdot n^4-6510\cdot n^3+17885\cdot n^2-23604\cdot n+16116)}{5040},n,1,12\right)$$

$\{1,2,4,7,12,20,33,62,147,394,1034,2509\}$

© next sequence: $\{1,-2,3,-4,5\}$. Let's make 19 is the next element!

$sq3:=\{1,-2,3,-4,5,19\}$ $\quad\{1,-2,3,-4,5,19\}$

$continue(sq3,10)$ $\quad\{1,-2,3,-4,5,19,-99,-699,-2431,-6332\}$

$rule(sq3)$ $$\frac{-(87\cdot n^5-1545\cdot n^4+10195\cdot n^3-30855\cdot n^2+42038\cdot n-20040)}{120}$$

$serie(sq3,10)$ $\quad\{1,-1,2,-2,3,22,-77,-776,-3207,-9539\}$

$rule\_series(sq3)$ $$\frac{-n\cdot(29\cdot n^5-531\cdot n^4+3625\cdot n^3-11405\cdot n^2+16266\cdot n-8224)}{240}$$

$$\mathrm{seq}\left(\frac{-n\cdot(29\cdot n^5-531\cdot n^4+3625\cdot n^3-11405\cdot n^2+16266\cdot n-8224)}{240},n,1,12\right)$$

$\{1,-1,2,-2,3,22,-77,-776,-3207,-9539,-23452,-50698\}$

The next page gives the Nspire code of the English version.

```
Define poss(sq)=
Func
:© returns number m of the first constant SoD
:Local i,m,term,auxsq
:m:=−1:i:=0
:auxsq:=sq
:While m=−1 and i≤20 and dim(auxsq)≥2
:  If konstant(auxsq) Then
:     m:=i
:     Else
:     i:=i+1
:     auxsq:=d_folge(1,auxsq)
:     EndIf
:     EndWhile
:     If m=−1 Then
:     m:=i
:     EndIf
:     Return m
:EndFunc
```

```
Define konstant(sq)=
Func
:Local i,element,truthval
:truthval:=true
:If dim(sq)=1 Then
:  Return false
:Else
:  element:=sq[1]
:  i:=2
:  While i≤dim(sq) and truthval=true
:      If sq[i]≠element Then
:      truthval:=false
:      EndIf
:      i:=i+1
:  EndWhile
:  Return truthval
:EndIf
:EndFunc
```

```
Define continue(sq,k)=
Func
:Local auxsq,i
:auxsq:={}
:For i,1,k
:auxsq:=augment(auxsq,{rule(sq)|n=i})
:EndFor
:Return auxsq
:EndFunc
```

```
Define rule(sq)=
Func
:Local i,m,term
:m:=poss(sq): term:=0
:  For i,0,m
:  term:=term+d_folge(i,sq)[1]*nCr(n-1,i)
:  EndFor
:  Return factor(term)
:EndFunc
```

```
Define d_seq(i,sq)=
Func
:If i=0 Then
:  Return sq
:  Else
:  Return d_seq(i-1,ΔList(sq))
:  EndIf
:EndFunc
```

File next_number.tns contains the English and German functions as well. File nächste_zahl.tns is Benno's first original version (in German only).

```
Define print(sq)=
Func
:Local i,k,n,matrix,sq_,anfang
:n:=dim(sq)-1:matrix:=newMat(n+1,2*dim(sq)-1)
:For i,1,n+1
:  For k,1,2*dim(sq)-1
:    matrix[i,k]:=_
:  EndFor
:EndFor
:For i,0,n
:  anfang:={}
:  For k,1,i
:    anfang:=augment({_},anfang)
:  EndFor
:  sq_:={}
:  For k,1,2*dim(d_folge(i,sq))-1
:    If mod(k,2)=1 Then
:    sq_:=augment(sq_,{d_folge(i,sq)[intDiv(k,2)+1]})
:    Else
:      sq_:=augment(sq_,{_})
:    EndIf
:  EndFor
:  sq_:=augment(anfang,sq_)
:  For k,1,dim(sq_)
:    matrix[i+1,k]:=sq_[k]
:  EndFor
:EndFor
:Return matrix
:EndFunc
```

```
Define serie(sq,k)=
Func
:Local sf,i
:sf:={0}
:For i,1,k
:sf:=augment(sf,{sf[i]+continue(sq,k)[i]})
:EndFor
:Return right(sf,dim(sf)-1)
:EndFunc
```

```
Define rule_series(sq)=
Func
:rule(serie(continue(sq,dim(sq)+1),dim(sq)+1))
:EndFunc
```

```
Define pascal(i,k)=
Func
:Local sq,n
:sq:={}
:For n,0,k
:  sq:=augment(sq,{nCr(n,i)})
:  EndFor
:  Return sq
:EndFunc
```

§ 62. **Arithmetische Reihen höherer Ordnung.**

1. Betrachten wir irgendeine Reihe A von Zahlen, die nach einem gegebenen Gesetz fortschreiten,

$$a_0,\ a_1,\ a_2,\ a_3,\ \ldots,$$

und bilden die Differenz je zweier aufeinander folgender Glieder

$$b_0 = a_1 - a_0,\quad b_1 = a_2 - a_1,\quad b_2 = a_3 - a_2,\ \ldots,$$

so heißt die Reihe B der b, also

$$b_0,\ b_1,\ b_2,\ \ldots,$$

die Reihe der Differenzen von der Reihe A.

Aus dieser können wir wieder die Differenzen bilden:

H. Weber, J. Wellstein; Encyklopädie der Elementar-Mathematik, I:Elementare Algebra und Analysis, 1909 Teubner, Leipzig.

https://archive.org/details/encyklomentmatik01weberich

Fractals and their Basin of Stability

David Halprin, Australia

Hans Lauwerier(*) outlined a method to predict the Cardioid basin arising from the iteration of the Circle in the Mandelbrot fractal, however he left it at the one example, therefore the generalisation of the approach begs investigation.

(1) $z_1 = F(z) = z^2 + c$

(2) $F(z) = F(z_0) + (z - z_0) \cdot F'(z_0)$

where $F(z_0)$ represents a translation, $(z - z_0)$ represents a rotation and $F'(z_0)$ represents a scale factor for enlargement, provided it is non-zero.

The iterative process depends on

(3) $z_{n+1} = F(z_n)$.

We have to consider the equilibrium points.

If, at a fixed point z, $|F'(z)| < 1$ then the orbit of z_0 in the vicinity of z approaches a stable (attracting) equilibrium point. If however $|F'(z)| > 1$ then the orbit is unstable and is repelled by z_0. The other case to be considered is when $|F'(z)| = 1$ where we have a neutral equilibrium.

We put $z_1 = z$ in (1) and solve

(4) $z = z^2 + c$

(5) $\alpha_1, \alpha_2 = -\frac{1}{2} \mp \sqrt{\frac{1}{4} - c}$

(6) $\alpha_1 + \alpha_2 = 1$

Graphically the mid point of the two roots is $\frac{1}{2}$. The stability of the orbit depends on

(7) $|F'(z)| = 2z$. Since $\alpha_2 > \frac{1}{2}$ it creates an unstable orbit so we discard it from further consideration, while $\alpha_1 < \frac{1}{2}$ creates a stable orbit.

At the margin of the area of stability, (the basin),

(8) $|2\alpha_1| = 1$

(9) $2\alpha_1 = \cos\theta + i \cdot \sin\theta = e^{i\theta} = 1 - \sqrt{1 - 4c}$

therefore

(10) $1 - 4c = 1 - 2e^{i\theta} + e^{2i\theta}$

(11) $c = \frac{1}{4}(e^{2i\theta} - 2e^{i\theta})$ Cardioids

(12) $a = -\frac{1}{4}(\cos 2\theta - 2\cos\theta)$; $b = -\frac{1}{4}(\sin 2\theta - 2\sin\theta)$

Mandelbrot, no doubt, was aware that the complex equation for a circle can be generalized for n, any positive integer exponent of z, yet all circles are concentric and coincident, with radius = 1: $f(z_1) = z^n + c$. His choice of n = 2 was the lowest value for n to produce a fractal, since n = 1 produces only a circular basin.

Jumping ahead, one finds that all the basins for the areas of stability for the total of (n – 1) fractals can be approximated to a high degree of accuracy by $c = k \cdot (e^{ni\theta} - n e^{i\theta})$, where '$k$' is an arbitrary constant. (They were generated by *DERIVE XM*, see .GIF-files below.)

Further, each of the basins has (n – 1) cusps starting with one cusp in the normal cardioid where n = 2. The background colour is contained within an (n – 1)-sided shape also, when iterated by FRACTINT, (see GIF-files below), a freeware software program, readily available for download from the web(**).

However the precise value for c can be calculated for n = 3,while the higher values of n seem to be insoluble in general terms.

(13) $$z_1 = F(z) = z^3 + c$$

(14) $$F'(z) = 3z^2$$

(15) $$z^3 + z + c = 0$$

(16) $$\alpha_1 = \frac{2}{\sqrt{3}} \cdot \sin\left(\frac{1}{3} \cdot \arcsin\frac{3\sqrt{3}c}{2}\right)$$

(17) $$\alpha_2 = -\frac{2}{\sqrt{3}} \cdot \sin\left(\frac{1}{3} \cdot \arcsin\frac{3\sqrt{3}c}{2}\right) + \frac{\pi}{3}$$

(18) $$\alpha_3 = \frac{2}{\sqrt{3}} \cdot \cos\left(\frac{1}{3} \cdot \arcsin\frac{3\sqrt{3}c}{2}\right) + \frac{\pi}{6}$$

(19) $$|F'(z)| = 3\alpha_1^2 = e^{i\theta} \therefore e^{\frac{i\theta}{2}} = 2 \cdot \sin\left(\frac{1}{3} \cdot \arcsin\frac{3\sqrt{3}\,c}{2}\right)$$

(20) $$\arcsin\left(\frac{e^{\frac{i\theta}{2}}}{2}\right) = \frac{1}{3} \cdot \arcsin\frac{3\sqrt{3}\ c}{2}$$

(21) $$c = \frac{2}{3\sqrt{3}} \cdot \sin\left(3 \cdot \arcsin\frac{e^{\frac{i\theta}{2}}}{2}\right)$$

(22) $$c = k \cdot \left(\frac{e^{\frac{i\theta}{2}}}{2} \cdot (3 - e^{i\theta}\right)$$

We take the first of the three solutions, since the other two do not produce stable orbits.

Equations (21) and (22) are alternative expressions for plotting the double cusped 'cardioid'.

(23) $F(z) = z^4 + c; \quad F'(z) = 4z^3$

(24) $z^4 - z + c = 0; \quad |F'(z)| = |4\alpha_1^{\ 3}| = e^{i\theta}$

(25) $c = k \cdot (e^{4i\theta} - 4e^{i\theta})$

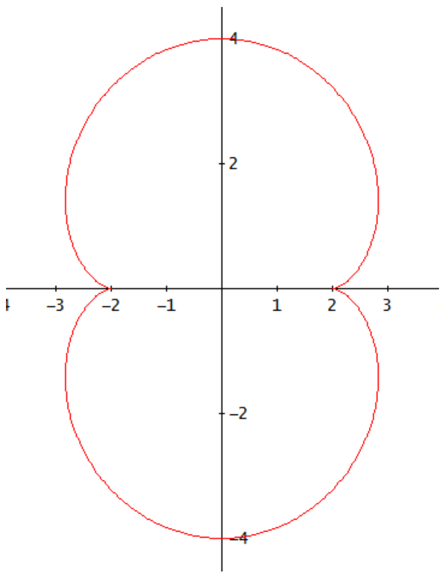
(26) $F(z) = z^n + c; \quad F'(z) = n \cdot z^{n-1}$

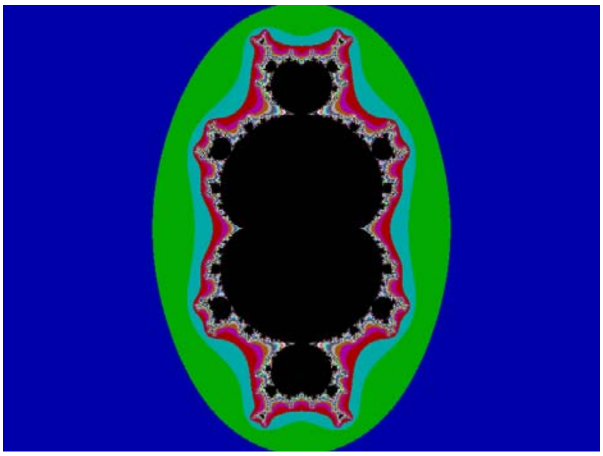
(27) $z^n - z + c = 0; \quad |F'(z)| = |n \cdot \alpha_1^{\ n-1}| = e^{i\theta}$

(28) $c = k \cdot (e^{ni\theta} - ne^{i\theta})$

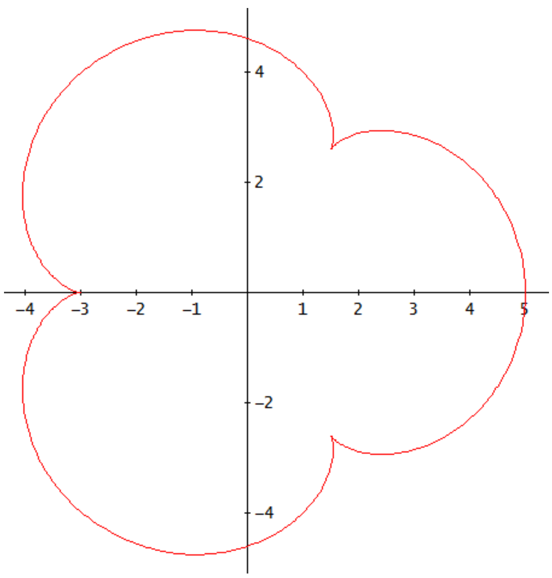
Equation (25) is taken from the general form, since the closed form solutions for *z* in terms of *c* cannot be solved exactly for *c* in terms of *z*.

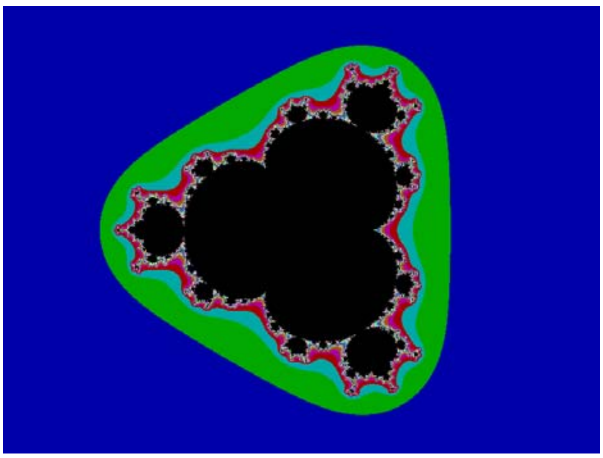
The reciprocal question then arises:- "an we nominate a basin of stability for a conjectured fractal and do the maths to find which curve(s), when iterated, will produce it?"



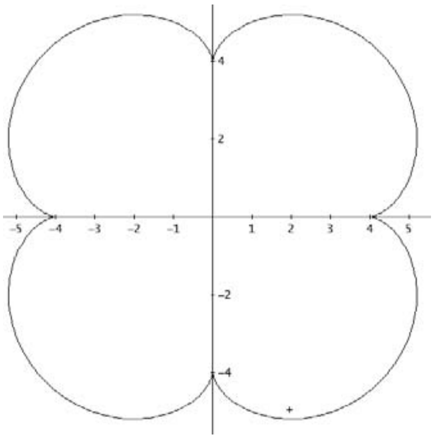


$\left[\mathrm{RE}\left(e^{3\cdot(i\cdot\theta)} - 3\cdot e^{i\cdot\theta}\right), \mathrm{IM}\left(e^{3\cdot(i\cdot\theta)} - 3\cdot e^{i\cdot\theta}\right)\right]$

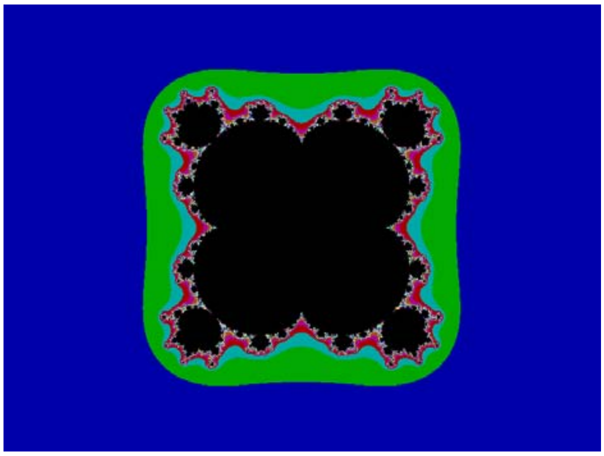


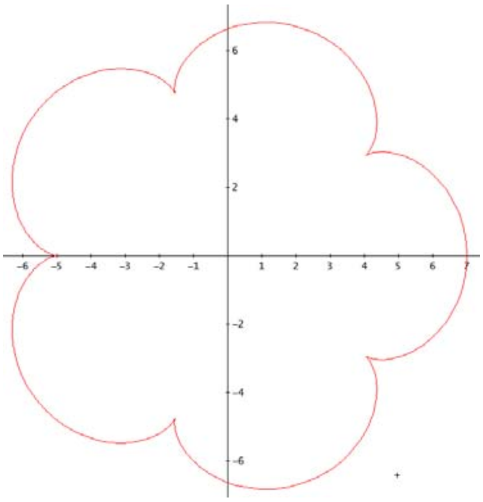


$\left[\mathrm{RE}\left(e^{4\cdot(i\cdot\theta)} - 4\cdot e^{i\cdot\theta}\right), \mathrm{IM}\left(e^{4\cdot(i\cdot\theta)} - 4\cdot e^{i\cdot\theta}\right)\right]$

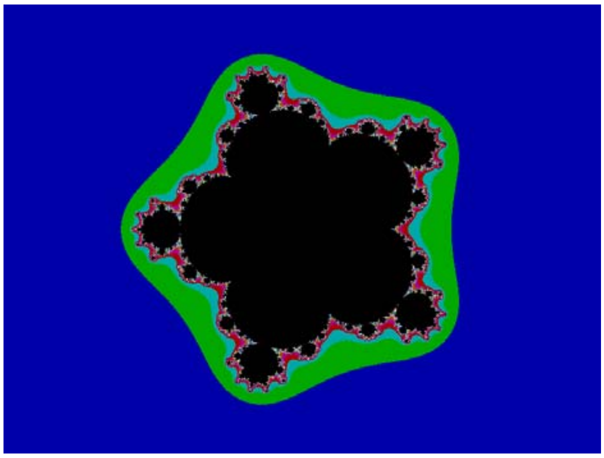


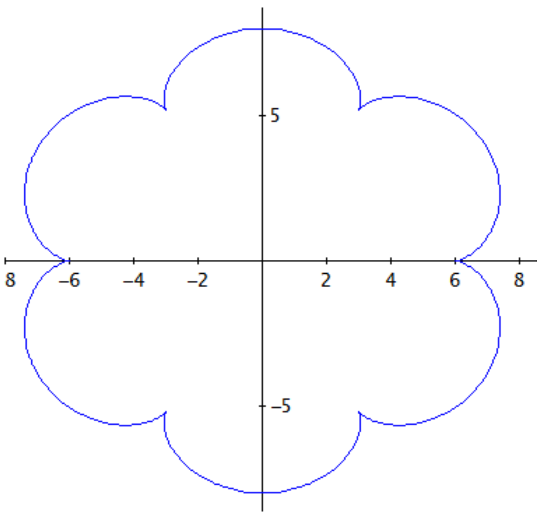
$\left[\mathrm{RE}\left(e^{5\cdot(i\cdot\theta)} - 5\cdot e^{i\cdot\theta}\right),\ \mathrm{IM}\left(e^{5\cdot(i\cdot\theta)} - 5\cdot e^{i\cdot\theta}\right)\right]$



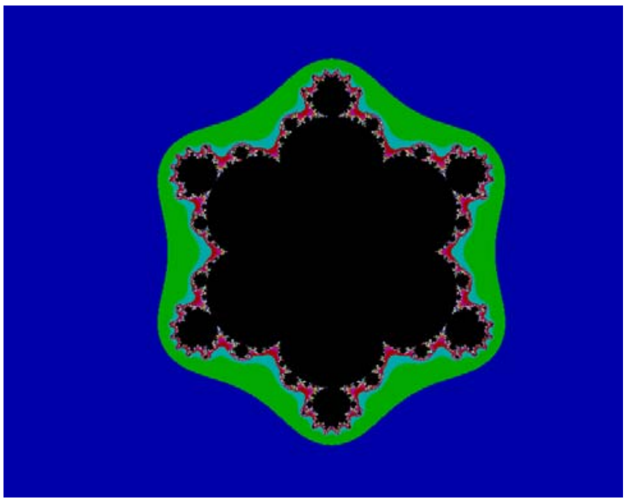


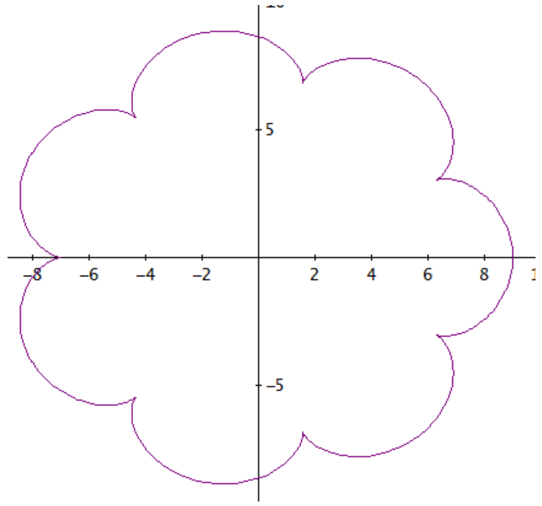
$\left[\mathrm{RE}\left(e^{6\cdot(i\cdot\theta)} - 6\cdot e^{i\cdot\theta}\right),\ \mathrm{IM}\left(e^{6\cdot(i\cdot\theta)} - 6\cdot e^{i\cdot\theta}\right)\right]$

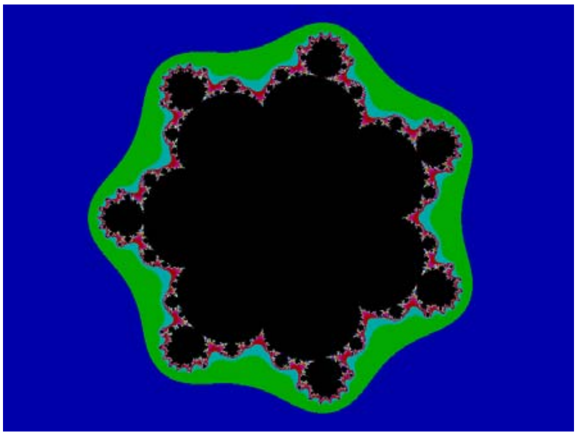




$\left[\mathrm{RE}\left(e^{7\cdot(i\cdot\theta)} - 7\cdot e^{i\cdot\theta}\right),\ \mathrm{IM}\left(e^{7\cdot(i\cdot\theta)} - 7\cdot e^{i\cdot\theta}\right)\right]$







$$\left[\mathrm{RE}\left(e^{8\cdot(i\cdot\theta)} - 8\cdot e^{i\cdot\theta}\right),\ \mathrm{IM}\left(e^{8\cdot(i\cdot\theta)} - 8\cdot e^{i\cdot\theta}\right)\right]$$

See the FRACTINT-Code to produce the last fractal:

```
Halp-Circle-10-8th {;David Halprin 37-1-10
     ; Yet to be analysed
     z = Pixel:          ;
     z = z * z * z * z * z * z * z * z
     z = z + Pixel,
     |z| <=4
}
```

(*)Hans Lauwerier books:

Fractals: Endlessly Repeated Geometrical Figures, ISBN-13: 978-0691024455

Fractals: Images of Chaos (Penguin Press Science), ISBN-13: 978-0140144116

Fraktale verstehen und selbst programmieren, Bd 1, 1989, Wittig Fachbuch, ISBN 3-88984-060-4

Fraktale verstehen und selbst programmieren, Bd 2, 1992, Wittig Fachbuch, ISBN 3-88984-061-2

(**) This is one of many websites where you can find FRACTINT for download:

(*) http://www.nahee.com/spanky/www/fractint/fractint.html

Two more memories of lovely Madeira

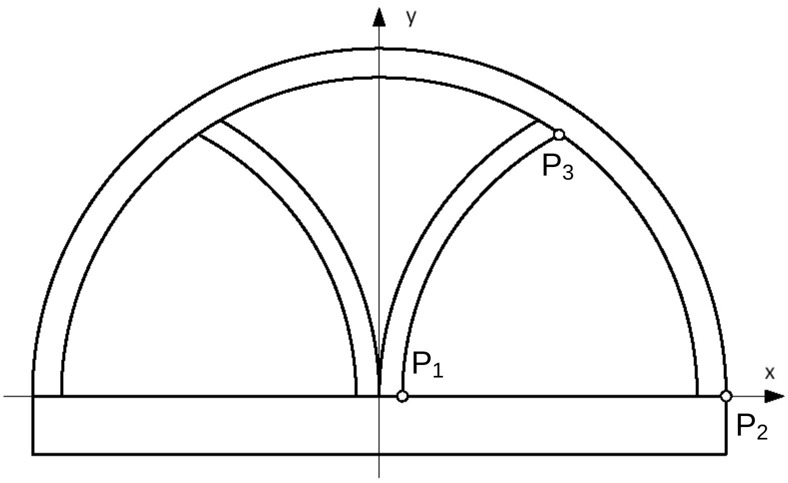
Inspiring Pavements in Funchal

Wolfgang Alvermann, Hinte, Germany

In Funchal, Madeira, one can find many interesting pavements on the streets and squares showing geometric patterns which might inspire for mathematical investigations.

One example is given by the picture on the left. The radius of the large semicircle is R = 2m, the rectangle on the base is 2t thick, and the difference of the radii (thickness of the arcs) is t = 0.15m.

Problem 1:



- Find the equations of all appearing semicircles and arcs in general form!
- Right hand side one can see a "gothic style" window formed by two arcs and a segment.
  Calculate the area of this "window". Calculate the coordinates of points P_1, P_2 and P_3 in general form. Then use the given measures for calculating the area.
- Calculate this area by integration, too, and compare the results!
- Produce a plot of one complete compartment of the pavement including the base rectangle.

The solution is given first in form of a TI-Nspire Notes page. Then we will present the respective DERIVE plot.

Solution

The next two screens present the solution starting with defining the circles, arcs and points.

Equations of circles and arcs

Upper semicircle: $x^2+y^2=r^2 \rightarrow$ $\mathbf{circ1}:=y=\sqrt{r^2-x^2}$ ▸ $y=\sqrt{r^2-x^2}$

Lower semicircle: $x^2+y^2=(r-t)^2 \rightarrow$ $\mathbf{circ2}:=y=\sqrt{(r-t)^2-x^2}$ ▸ $y=\sqrt{r^2-2\cdot r\cdot t+t^2-x^2}$

Right "Gothic Window"

Upper arc: $(x-r)^2+y^2=r^2 \rightarrow$ $\mathbf{ruparc}:=y=\sqrt{r^2-(x-r)^2}$ ▸ $y=\sqrt{(2\cdot r-x)\cdot x}$

Lower arc: $(x-r)^2+y^2=(r-t)^2 \rightarrow$ $\mathbf{rloarc}:=y=\sqrt{(r-t)^2-(x-r)^2}$ ▸ $y=\sqrt{-(2\cdot r-t-x)\cdot(t-x)}$

Left "Gothic Window"

Upper arc: $(x+r)^2+y^2=r^2 \rightarrow$ $\mathbf{luparc}$ ▸ $y=\sqrt{-(2\cdot r-t+x)\cdot(t+x)}$

Lower arc: $(x+r)^2+y^2=(r-t)^2 \rightarrow$ $\mathbf{luparc}:=y=\sqrt{(r-t)^2-(x+r)^2}$ ▸ $y=\sqrt{-(2\cdot r-t+x)\cdot(t+x)}$

Points

P_1: $\mathbf{p1}:=[r\ \ 0]$ ▸ $[r\ \ 0]$ P_2: $\mathbf{p2}:=[t\ \ 0]$ ▸ $[t\ \ 0]$

P_3: $\mathbf{p3}:=\left[\frac{r}{2}\ \ \text{right}(\mathbf{rloarc})\right]|x=\frac{r}{2}$ ▸ $\left[\frac{r}{2}\ \ \frac{\sqrt{(r-2\cdot t)\cdot(3\cdot r-2\cdot t)}}{2}\right]$ $\mathbf{p3}|r=2 \text{ and } t=0.15$ ▸ $[1\ \ 1.55644]$

We calculate α = $\triangle P_2P_1P_3$ in order to calculate the area of the segment of the circle formed by the arc and the segment P_2P_3. Two segments and the area of the triangle $\Delta P_2P_1P_3$ will give the requested area.

$\boldsymbol{\alpha}:=\tan^{-1}\left(\frac{\mathbf{p3}[1,2]}{\frac{r}{2}}\right)|r=2 \text{ and } t=0.15$ ▸ 0.999717

Area of the segment of the circle: $\mathbf{a1}:=\frac{(r-t)^2}{2}\cdot(\boldsymbol{\alpha}-\sin(\boldsymbol{\alpha}))|r=2 \text{ and } t=0.15$ ▸ 0.27106

Area of the triangle: $\mathbf{a2}:=\frac{1}{2}\cdot(r-2\cdot t)\cdot\mathbf{p3}[1,2]|r=2 \text{ and } t=0.15$ ▸ 1.32297

Area of the Window: $2\cdot\mathbf{a1}+\mathbf{a2}$ ▸ 1.86509

And by integration: $2\cdot\int_t^{\frac{r}{2}}\text{right}(\mathbf{rloarc})\,dx|r=2 \text{ and } t=0.15$ ▸ 1.86509

This is one way to plot the pattern: we define the function – some of them piecewise. It seems to be necessary first fixing the functions and then denoting them from f1 to f10

$\text{right}(\textbf{circ1})|r=2 \blacktriangleright \sqrt{4-x^2}$ $\quad \text{f1}(x):=\sqrt{4-x^2} \blacktriangleright \textit{Done}$

$\text{right}(\textbf{circ2})|r=2 \text{ and } t=0.15 \blacktriangleright \sqrt{3.4225-x^2}$ $\quad \text{f2}(x):=\sqrt{3.4225-x^2} \blacktriangleright \textit{Done}$

$\text{right}(\textbf{ruparc})|r=2 \text{ and } 0\le x\le 0.856 \blacktriangleright \sqrt{-x\cdot(x-4)}$ $\quad \text{f3}(x):=\sqrt{-x\cdot(x-4)} \blacktriangleright \textit{Done}$

$\text{right}(\textbf{rloarc})|r=2 \text{ and } t=0.15 \blacktriangleright \sqrt{-x^2+4.\cdot x-0.5775}$ $\quad \text{f4}(x):=\sqrt{-x^2+4.\cdot x-0.5775} \blacktriangleright \textit{Done}$

$\text{right}(\textbf{luparc})|r=2 \text{ and } -0.856\le x\le 0 \blacktriangleright \sqrt{-x\cdot(x+4)}$ $\quad \text{f5}(x):=\sqrt{-x\cdot(x+4)}$

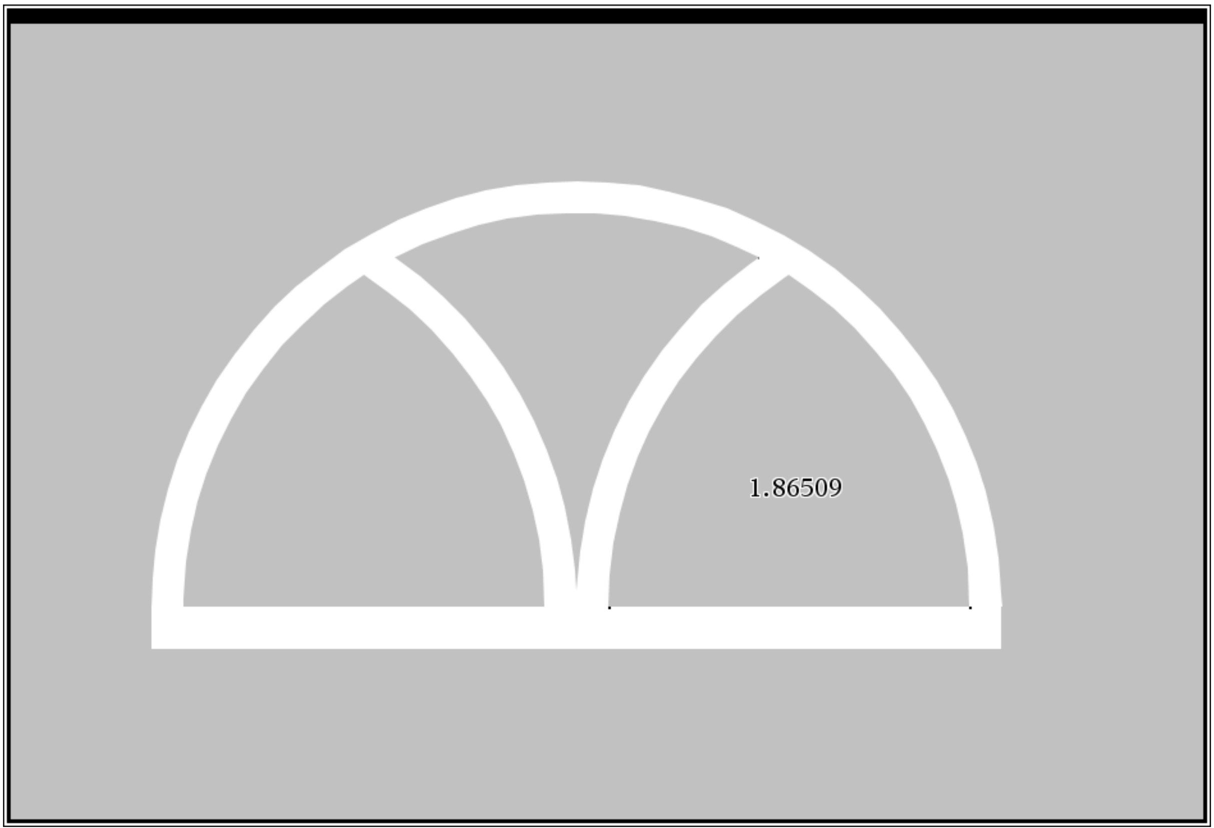
$\text{right}(\textbf{lloarc})|r=2 \text{ and } t=0.15 \blacktriangleright \sqrt{-x^2-4.\cdot x-0.5775}$ $\quad \text{f6}(x):=\sqrt{-x^2-4.\cdot x-0.5775} \blacktriangleright \textit{Done}$

$\text{f7}(x):=\begin{cases}\text{f2}(x), & x\le -1\\ \text{f6}(x), & x>-1\end{cases} \blacktriangleright \textit{Done}$ $\quad \text{f9}(x):=\begin{cases}\text{f4}(x), & x\le 1\\ \text{f2}(x), & x>1\end{cases} \blacktriangleright \textit{Done}$

$\text{f8}(x):=\text{f2}(x)|-0.856\le x\le 0.856 \blacktriangleright \textit{Done}$

$\text{f10}(x):=\begin{cases}\text{f5}(x), & 0\ge x\ge -0.856\\ \text{f3}(x), & 0\le x\le 0.856\end{cases} \blacktriangleright \textit{Done}$

We colour the pavement using the "Integral" and "Bounded Area" tool contained in the "Analyze Graph" menu. All labels are hidden except one: the area of the window.



We use DERIVE to plot the inverse image. (This might be an additional task for the Nspire users, too. Making the picture from an appropriate position it might serve as background for Nspire and DERIVE as well.)

#1: $[r := 2,\ t := 0.15]$

#2: $\left[y1(x) := \sqrt{(r^2 - x^2)},\ y2(x) := \sqrt{((r - t)^2 - x^2)}\right]$

#3: $\left[y3(x) := \sqrt{((r - t)^2 - (x + r)^2)},\ y4(x) := \sqrt{(r^2 - (x + r)^2)}\right]$

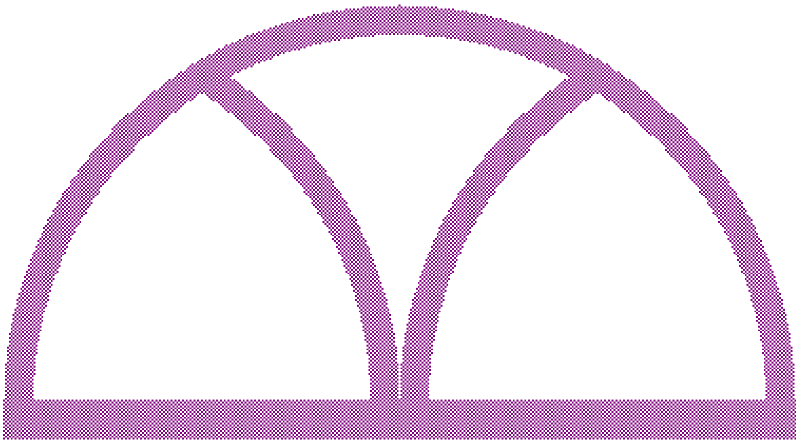
#4: $\left[y5(x) := \sqrt{((r - t)^2 - (x - r)^2)},\ y6(x) := \sqrt{(r^2 - (x - r)^2)}\right]$

#5: $y2(x) \le y \le y1(x) \vee (0 \le y \le y1(x) \wedge 1.85 \le |x| \le 2)$

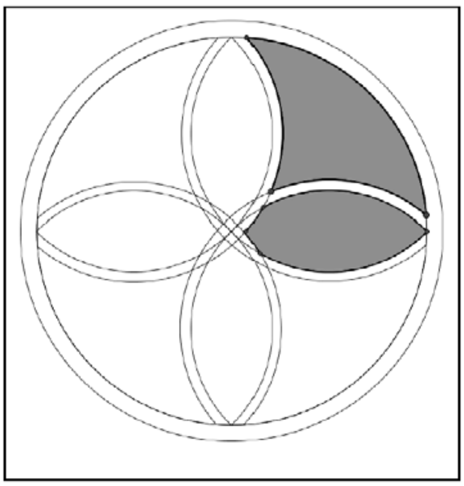
#6: $(0 \le y \le y4(x) \wedge -0.15 < x) \vee (y3(x) \le y \le y4(x) \wedge x \ge -1)$

#7: $(0 \le y \le y6(x) \wedge x < 0.15) \vee (y5(x) < y < y6(x) \wedge x \le 1)$

#8: $-2 \le x \le 2 \wedge -0.2 \le y \le 0$



Problem 2:



Another pattern from Funchal: the regions which are bounded by bold curves are filled with dark paving stones, the others with white ones.

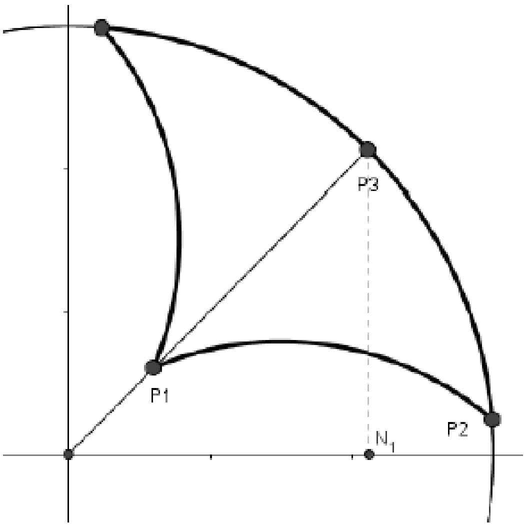
The radius of the inner circle is r = 3m, the radius of the outer one is r + 2t with t = 0.1m.

The arcs forming the inner four leaved rosette are t thick.

Problem 2 Part a:

- Make a sketch of the figure at a scale of 1:50. What are the centres of the arcs? You are invited to plot the figure on a TI-Nspire Graph page or with DERIVE (or any other tool, e.g. Geogebra, …!
- The grey area in the figure above is to be calculated applying appropriate means. It is necessary to find the equations of the respective circles!

Solution



Due to symmetry we need only to calculate the area described by the points P_1, P_2 and P_3.

Line y = x intersects the circle in P_3. N_1 is the pedal point of the verticular line through P_3. P_3N_1 splits the region into two parts. The area of the two parts can be calculated by means of integration.

We need the equation of the circles in order to find the coordinates of the intersection points.

Then we will integrate.

circle1: centre = [0,0]; r

$\textbf{circ1}:=y=\sqrt{r^2-x^2} \;\blacktriangleright\; y=\sqrt{r^2-x^2}$

circle2: centre = [r/2,−r/2]; r = r/2*$\sqrt{2}$ +t

$\text{solve}\left(\left(x-\frac{r}{2}\right)^2+\left(y+\frac{r}{2}\right)^2=\left(\frac{r}{2}\cdot\sqrt{2}+t\right)^2,y\right)$

$\blacktriangleright\; y=\frac{\sqrt{r^2+4\cdot r\cdot(\sqrt{2}\cdot t+x)+4\cdot(t^2-x^2)}-r}{2} \text{ or } y=\frac{-\left(\sqrt{r^2+4\cdot r\cdot(\sqrt{2}\cdot t+x)+4\cdot(t^2-x^2)}+r\right)}{2}$

$\text{circ2}:=y=\frac{\sqrt{r^2+4\cdot r\cdot(\sqrt{2}\cdot t+x)+4\cdot(t^2-x^2)}-r}{2} \;\blacktriangleright\; y=\frac{\sqrt{r^2+4\cdot r\cdot(\sqrt{2}\cdot t+x)+4\cdot(t^2-x^2)}-r}{2}$

Point P1:

$\text{solve}(\textbf{circ2} \text{ and } y=x,\{x,y\})|r=3 \text{ and } t=0.1$

$\blacktriangleright\; x=^-0.465974 \text{ and } y=^-0.465974 \text{ or } x=0.465974 \text{ and } y=0.465974$

$\textbf{p1}:=[0.465974 \quad 0.465974] \;\blacktriangleright\; [0.465974 \quad 0.465974]$

Point P2:

$\text{solve}(\textbf{circ1} \text{ and } \textbf{circ2},\{x,y\})|r=3 \text{ and } t=0.1 \;\blacktriangleright\; x=2.99666 \text{ and } y=0.14142$

$\textbf{p2}:=[2.99666 \quad 0.14142] \;\blacktriangleright\; [2.99666 \quad 0.14142]$

Point P3:

$\text{solve}(\mathbf{circ1} \text{ and } y=x,\{x,y\})|r=3 \text{ and } t=0.1 \blacktriangleright x=\frac{3\cdot\sqrt{2}}{2} \text{ and } y=\frac{3\cdot\sqrt{2}}{2}$

$\mathbf{p3}:=\left[\frac{\sqrt{2}\cdot 3}{2} \quad \frac{\sqrt{2}\cdot 3}{2}\right] \blacktriangleright [2.12132 \quad 2.12132]$

Left part:

$\mathbf{a1}:=\int_{0.465974}^{2.12132} (x-\text{right}(\mathbf{circ2}))\,dx|r=3 \text{ and } t=0.1 \blacktriangleright 1.0515$

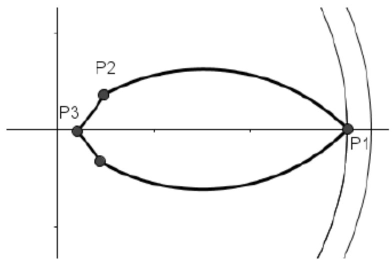
$\mathbf{a2}:=\int_{2.12132}^{2.99666} (\text{right}(\mathbf{circ1})-\text{right}(\mathbf{circ2}))\,dx|r=3 \text{ and } t=0.1 \blacktriangleright 0.906648$

Total area = $2\cdot(\mathbf{a1}+\mathbf{a2}) \blacktriangleright 3.91629$

Additional task: Try to find the area without applying integration!

Problem 2 Part b:

- Calculate the area marked by bold curves as shown below.



We need two circles to calculate the intersection points P_1, P_2 and P_3

We apply integration again.

circle 1: centre = [r/2,−r/2]; r/2*$\sqrt{2}$

$\text{solve}\left(\left(x-\frac{r}{2}\right)^2+\left(y+\frac{r}{2}\right)^2=\left(\frac{r\cdot\sqrt{2}}{2}\right)^2,y\right) \blacktriangleright y=\frac{\sqrt{r^2+4\cdot r\cdot x-4\cdot x^2}-r}{2} \text{ or } y=\frac{-\left(\sqrt{r^2+4\cdot r\cdot x-4\cdot x^2}+r\right)}{2}$

$\mathbf{circ1}:=y=\frac{\sqrt{r^2+4\cdot r\cdot x-4\cdot x^2}-r}{2} \blacktriangleright y=\frac{\sqrt{r^2+4\cdot r\cdot x-4\cdot x^2}-r}{2}$

circle 2: centre = [−r/2,r/2]; r/2*$\sqrt{2}$ +t

$\text{solve}\left(\left(x+\frac{r}{2}\right)^2+\left(y-\frac{r}{2}\right)^2=\left(\frac{r}{2}\cdot\sqrt{2}+t\right)^2,y\right)$

$\blacktriangleright y=\frac{-\left(\sqrt{r^2+4\cdot r\cdot\left(\sqrt{2}\cdot t-x\right)+4\cdot\left(t^2-x^2\right)}-r\right)}{2} \text{ or } y=\frac{\sqrt{r^2+4\cdot r\cdot\left(\sqrt{2}\cdot t-x\right)+4\cdot\left(t^2-x^2\right)}+r}{2}$

$\mathbf{circ2}:=y=\frac{-\left(\sqrt{r^2+4\cdot r\cdot\left(\sqrt{2}\cdot t-x\right)+4\cdot\left(t^2-x^2\right)}-r\right)}{2} \blacktriangleright y=\frac{-\left(\sqrt{r^2+4\cdot r\cdot\left(\sqrt{2}\cdot t-x\right)+4\cdot\left(t^2-x^2\right)}-r\right)}{2}$

At first calculate points P_1, P_2 and P_3. Then it is easy to find the requested area as the double sum of two integrals:

Point P_1: $\mathbf{p1}:=[3\ \ 0] ▸ [3\ \ 0]$

Point P_2:

$\text{solve}(\mathbf{circ1} \text{ and } \mathbf{circ2},\{x,y\})|r=3 \text{ and } t=0.1 \text{ and } x>0 ▸ x=0.363689 \text{ and } y=0.291312$

$\mathbf{p2}:=[0.363689\ \ 0.291312] ▸ [0.363689\ \ 0.291312]$

Point P_3:

$\text{zeros}(\text{right}(\mathbf{circ2}),x)|r=3 \text{ and } t=0.1 \text{ and } x>0 ▸ \{0.138372\}$

$\mathbf{p3}:=[0.138372,0] ▸ [0.138372\ \ 0]$

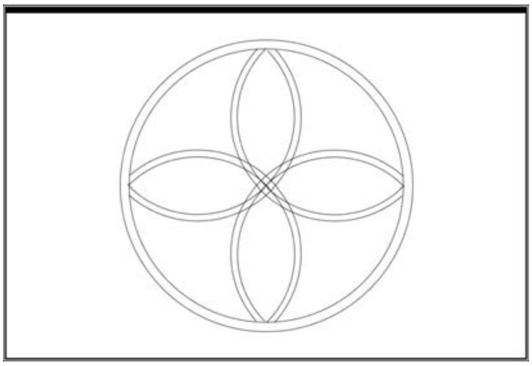
Area =

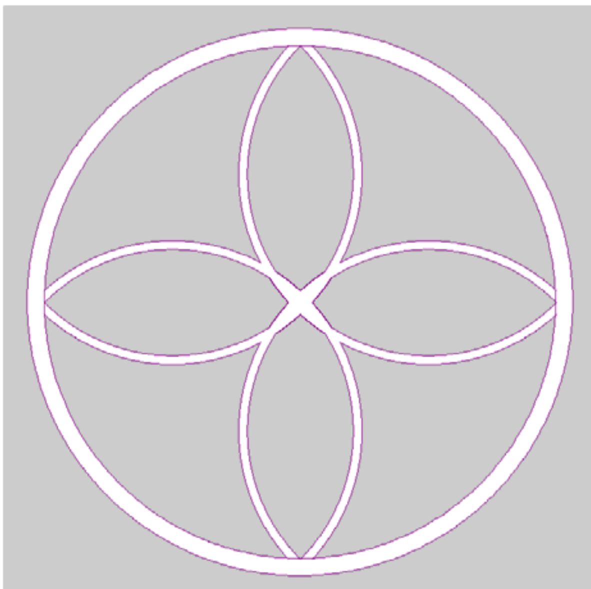
$$2\cdot\left(\int_{\mathbf{p3}[1,1]}^{\mathbf{p2}[1,1]} \text{right}(\mathbf{circ2})\, dx+\int_{\mathbf{p2}[1,1]}^{3} \text{right}(\mathbf{circ1})\, dx | r=3 \text{ and } t=0.1\right) ▸ \mathbf{2.51654}$$

Josef's additional comments:

The screen below shows the TI-Nspire Graph page with the plot of the pattern of the pavement.

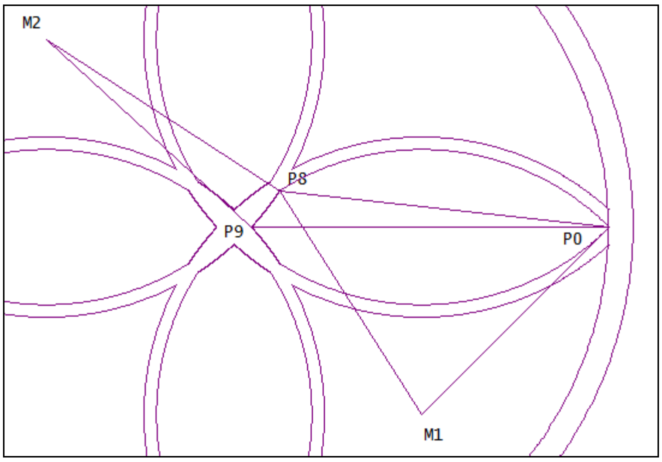
It was produced without defining any functions, i.e. applying geometry tools only (Circle, Intersection point, Circle arc, Reflection).





The right figure is the DERIVE plot. This was very much more difficult – but also much more interesting – to create. It is a good exercise for working with angles and parameter representation of circles (to define the many appearing circle arcs). The grey colour was brought in by first converting the plot to a Bitmap Image and then filling the regions with Paint tools. All is done within the DERIVE environment, see more in DNL#63.

For all of you who need more stuff for the students, here is an additional task:



Find the area without integration!

Hint:

2*(triangle P9P8P0 + segment P0P8 – segment P9P8)

The coordinates of the points were precalculated by intersecting the respective circles. (See also the Nspire results from above.)

#65: [m1 = [1.5, -1.5], m2 = [-1.5, 1.5]]

#66: [p0 = [3, 0], p8 = [0.363688897, 0.2913115521], p9 = [0.138372384, 0]]

#67: $[r1 := 1.5\cdot\sqrt{2},\ r2 := 1.5\cdot\sqrt{2} + 0.1]$

#68: $$\text{angle}(x, y, z) := \text{ATAN}\left(\frac{\dfrac{x_2 - y_2}{x_1 - y_1} - \dfrac{z_2 - y_2}{z_1 - y_1}}{1 + \dfrac{x_2 - y_2}{x_1 - y_1}\cdot\dfrac{z_2 - y_2}{z_1 - y_1}}\right)$$

#69: angle(p8, m1, [3, 0]) = 1.350689900

#70: angle(p8, m2, p9) = 0.1659836335

Δ P9P0P8

#71: $$\frac{(3 - 0.13837)\cdot 0.29131}{2} = 0.4168107176$$

Segment P0P8

#72: $$\frac{r1^2}{2}\cdot(1.3506899 - \text{SIN}(1.3506899)) = 0.8433352837$$

Segment P9P8

#73: $$\frac{r2^2}{2}\cdot(0.1659836335 - \text{SIN}(0.1659836335)) = 0.001877753847$$

Area of the drop formed region:

Fortunately we obtain the same results!

#74: 2·(0.4168107176 + 0.8433352837 – 0.001877753847) = 2.516536494